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Agreements with overlapping coalitions

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À ma mère Lydia et à la mémoire de mon père Crespin

« *Tout ce que tu désires ardemment, la nature conspire à te l'accorder* »

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Résumé

Dans certaines circonstances, des actions de groupes sont plus performantes que des actions individuelles. Dans ces situations, il est préférable de former des coalitions. Ces coalitions peuvent être disjointes ou imbriquées. La littérature économique met un fort accent sur la modélisation des accords où les coalitions d'agents économiques sont des ensembles disjoints. Cependant on observe dans la vie de tous les jours que les coalitions politiques, environnementales, de libre-échange et d'assurance informelles sont la plupart du temps imbriquées. Aussi, devient-il impératif de comprendre le fonctionnement économique des coalitions imbriquées. Ma thèse développe un cadre d'analyse qui permet de comprendre la formation et la performance des coalitions même si elles sont imbriquées.

Dans le premier chapitre je développe un jeu de négociation qui permet la formation de coalitions imbriquées. Je montre que ce jeu admet un équilibre et je développe un algorithme pour calculer les allocations d'équilibre pour les jeux symétriques. Je montre que toute structure de réseau peut se décomposer de manière unique en une structure de coalitions imbriquées. Sous certaines conditions, je montre que cette structure correspond à une structure d'équilibre d'un jeu sous-jacent.

Dans le deuxième chapitre j'introduis une nouvelle notion de noyau dans le cas où les coalitions imbriquées sont permises. Je montre que cette notion de noyau est une généralisation naturelle de la notion de noyau de structure de coalitions. Je vais plus loin en introduisant des agents plus raffinés. J'obtiens alors le noyau de structure de coalitions imbriquées que je montre être un affinement de la première notion.

Dans la suite de la thèse, j'applique les théories développées dans les deux premiers chapitres à des cas concrets.

Le troisième chapitre est une application de la relation biunivoque établie dans le premier chapitre entre la formation des coalitions et la formation de réseaux. Je propose une modélisation réaliste et effective des assurances informelles. J'introduis ainsi dans la littérature économique sur les assurances informelles, quatre innovations majeures : une fusion entre l'approche par les groupes et l'approche par les réseaux sociaux, la possibilité d'avoir des organisations imbriquées d'assurance informelle, un schéma de punition endogène et enfin les externalités. Je caractérise les accords d'assurances informelles stables et j'isole les conditions qui poussent les agents à dévier. Il est admis dans la littérature que seuls les individus ayant un revenu élevé peuvent se permettre de violer les accords d'assurances informelles. Je donne ici les conditions dans lesquelles cette hypothèse tient. Cependant, je montre aussi qu'il est possible de violer cette hypothèse sous d'autres conditions réalistes. Finalement je dérive des résultats de statiques comparées sous deux normes de partage différents.

Dans le quatrième et dernier chapitre, je propose un modèle d'assurance informelle où les groupes homogènes sont construits sur la base de relations de confiance préexistantes. Ces groupes sont imbriqués et représentent des ensembles de partage de risque. Cette approche est plus générale que les approches traditionnelles de groupe ou de réseau. Je caractérise les accords stables sans faire d'hypothèses sur le taux d'escompte. J'identifie les caractéristiques des réseaux stables qui correspondent aux taux d'escomptes les plus faibles. Bien que l'objectif des assurances informelles soit de lisser la consommation, je montre que des effets externes liés notamment à la valorisation des liens interpersonnels renforcent la stabilité. Je développe un algorithme à pas finis qui égalise la consommation pour tous les individus liés. Le fait que le nombre de pas soit fini (contrairement aux algorithmes à pas infinis existants) fait que mon algorithme peut inspirer de manière réaliste des politiques économiques. Enfin, je donne des résultats de statique comparée pour certaines valeurs exogènes du modèle.

Mots-clés : coalitions imbriquées, réseaux, jeux de fonctions de couverture, négociation, noyau, stabilité assurance informelle, normes de partage.

Abstract

When groups can perform a task more efficiently than single individuals, a desirable behavior is to form coalitions. Coalitions can be disjoint or overlapping. But, almost all the economics literature on coalition formation is mostly restricted to models where coalitions are disjoint. However, in politics, environmental issues, customs unions, informal insurance, and many other economic, social, and political interactions, we observe overlapping coalitions. How can we understand agreements involving overlapping coalitions ? How can we study their efficiency if there is no theory to model them ? My thesis solves these questions by developing a framework on coalition formation that accommodates overlapping coalitions.

In the first chapter, following a non-cooperative approach, I develop a bargaining game to model the formation of overlapping coalitions. I show the existence of a subgame perfect equilibrium and I provide an algorithm that generates equilibrium outcomes for symmetric games. I establish an overlapping coalition's representation for each network and I show that, under some conditions, they are equilibrium outcomes.

The core is the most popular solution concept in cooperative game. In the second chapter, following a cooperative approach, I develop a new concept of core that accommodates overlapping coalitions, and coincides with the recursive core when coalitions do not intersect. First, I extend naively the residual game to only embody overlapping coalitions, and I obtain a range of allocations between the optimistic and the pessimistic core. Secondly, I provide a consistent notion of residual game and I show

that the overlapping coalition structure core not only stands as a generalization of the coalition structure core, but also induces a refinement of the extended recursive core.

As I build a theoretical framework for overlapping coalition formation, I provide applications in the remainder of the thesis.

In the third chapter, I model informal insurance arrangements as a collection of overlapping trust coalitions. The model is based on empirical facts. I enrich the theoretical modeling of informal insurance arrangements by introducing four key features : the merging of the group approach and the network approach, the possibility for informal insurance organizations to overlap, the endogenous punishment scheme, and externalities. I characterize self enforcing stable informal insurance arrangements and I derive conditions under which deviation occurs. While it is always assumed in the literature that only wealthy individuals may deviate. I formally isolate conditions under which this assumption holds. Furthermore, I show that if these conditions does not hold, this assumption is violated. Finally, I provide static comparative results for consumption under two distinct sharing norms.

Finally in the fourth chapter, I investigate multilateral informal insurance organizations built on networks of trust relationships. The model is based on empirical findings and nests the traditional approaches which use bilateral links or groups. I characterize self enforcing stable informal insurance organizations without imposing extreme discounting. I show that density and clustering characterize networks that match the lowest discounting for stability. While insurance is formally arranged to smooth consumption, I show that external effects such as social privileges tend to reinforce the stability of such arrangement. I use my stability results to derive comparative statics for exogenous parameters of the model. Finally, I develop a procedure in finite steps that equates consumption for all linked individuals. Contrary to the existing procedures in infinite steps, my procedure is more realistic and useful to policy makers.

Keywords : overlapping coalitions, networks, cover function games, bargaining, core, stability, informal insurance, sharing norms

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Introduction générale

Depuis les années 90, on observe une prolifération des zones de libre-échange à travers les cinq continents. L'Organisation Mondiale du Commence en a recensé 150 en 2012. Il s'agit de groupes de pays qui s'entendent pour éliminer les droits de douane et autres restrictions à l'importation. Vu d'un point de vue économique, il s'agit de coalitions formées par des pays qui s'entendent pour améliorer leurs rendements commerciaux. On observe également des coalitions dans divers situations économiques : accords environnementaux, groupes d'assurances informelles, groupes de recherche, groupes humanitaires, groupes de fourniture de biens publics etc. Ces coalitions peuvent être des ensembles disjoints ou bien souvent des ensembles imbriqués.
La majeure partie de la littérature économique consacrée à la formation des coalitions, se focalise sur les coalitions disjointes. Cependant force est de constater que les théories développées dans ce cas ne s'appliquent généralement pas aux coalitions imbriquées. De surcroît, la présence accrue de coalitions imbriquées dans les regroupements économiques poussent des chercheurs (Ray, 2007) à plaider pour l'urgence de leur prise en compte dans la théorie économique. Ma thèse se propose donc d'établir un cadre d'analyse qui s'applique à toutes les formes de coalitions, aussi bien disjointes qu'imbriquées. Pour ce faire, la thèse se subdivise en quatre chapitres. Les deux premiers chapitres proposent des éléments de réponse aux questions liées à la formation même des coalitions imbriquées. Il s'agit de comprendre par quels processus ces genres de coalitions voient le jour. Forts de ces éléments de réponses, les deux

derniers chapitres s'attaquent aux assurances informelles (qui sont souvent des coalitions imbriquées) en tentant de les analyser à la lumière des réponses préalablement apportées.

Dans le premier chapitre, je me pose la question suivante : peut-on par un jeu dynamique de formation de coalitions aboutir à un équilibre où les coalitions qui se forment sont imbriquées ? Pour y répondre, je propose un jeu de négociation où des propositions de formation de coalitions et de partage de surplus réalisés sont faites par étapes. Je montre que ce jeu admet un équilibre parfait en sous-jeux. Une fois cette étape franchie, la question la plus important est de savoir s'il est possible de calculer concrètement cet équilibre et sous quelles conditions cela est possible. Je propose donc une procédure qui permet, sous certaines conditions de régularité, de calculer cet équilibre. Cette procédure se limite toutefois aux jeux symétriques : cas où le nom des individus n'est pas important mais seul la structure des groupes auxquels ils appartiennent compte. Dans la suite j'établis un lien biunivoque entre les coalitions imbriquées et les réseaux interrelationnels. Ce lien me permet de caractériser tous les réseaux interrelationnels par des coalitions imbriquées et inversement de dégager les formes de coalitions imbriquées qui peuvent être caractérisés par des réseaux. Partant de là, je montre que pour la plupart des réseaux (excepté les cercles où chaque agent a deux liens et tous les agents sont liés), il existe un jeu dynamique qui admet ce réseau comme un émanant d'un équilibre parfait en sous-jeux (du jeu que je propose).

Dans le second chapitre, toujours dans le souci de proposer un processus rationnel aboutissant à la formation de coalitions imbriquées, je propose une approche alternative. Il s'agit d'une approche qui s'inscrit dans la tradition des jeux de coopération, où des agents peuvent refuser la coopération proposée et former leur propre coopération, tant que cette déviation leur permet d'améliorer leurs gains. Je m'intéresse notamment à une structure particulière de distribution de gains : le noyau. Il s'agit des distributions qui ne peuvent être dominées : on ne peut pas faire strictement mieux pour les agents. Il s'agit donc de regarder comment cette notion de noyau peut être étendue à un cas où la formation de coalitions imbriquées est possible. Je propose des extensions du noyau récursif préalablement proposé dans la littérature mais qui ne s'applique qu'aux coalitions disjoints. Je montre qu'une extension naïve (qui s'ouvre aux coalitions imbriquées tout en conservant la modélisation existante de la réaction suite à une déviation) peut aboutir à une structure de noyau pour les coalitions imbriquées. Cette structure présente cependant beaucoup de limites et ne modélise pas toutes les formes possibles de coalitions imbriquées. Dans la suite du chapitre, je propose une approche beaucoup plus adaptée aux coalitions imbriquées en insistant sur le rôle des agents qui sont membres de plusieurs coalitions. La notion de noyau ainsi obtenue présente une structure beaucoup plus fine que la première et semble mieux adaptée aux coalitions imbriquées.

Dans le troisième chapitre, j'utilise le lien établi entre les réseaux interrelationnels et les coalitions imbriquées pour expliquer dans un premier temps la structure imbriquées des assurances informelles. Il s'agit d'une structure organisationnelle où des groupes d'agents économiques, face à l'indisponibilité de crédit ou d'assurance, s'associent de manière non formellement réglementée pour se faire réciproquement des transferts dans le souci de lisser leur consommation. Ce chapitre vise essentiellement à caractériser les déterminants de la stabilité de ces groupes de transfert. Par stabilité j'entends la non-possibilité de défection par certains agents. Pour soutenir cette non-défection, souvent dans le secteur formel, il y a le système judicaire ou les agences de recouvrement. Mais dans les économies agraires où ces structures formelles font défaut, les groupes d'assurances développent des techniques d'autodiscipline. Souvent ces techniques sont supposées exogènes dans la littérature. Je propose une procédure d'autodiscipline endogène où les agents n'obéissent pas à une règle préétablie pour sanctionner ceux qui font défection, mais agissent plutôt de manière à maximiser leur bien-être. En outre, j'introduis des externalités (les actions au sein d'un groupe influencent les gains des autres agents, incluant même les non-membres de ce groupe), largement documentées dans la littérature empirique, dans la construction du modèle.

En explorant deux règles de partage, je caractérise les groupes de transfert stables. J'isole ensuite les conditions qui favorisent les défections et j'identifie le rôle des externalités dans la stabilité. Enfin, je dérive des résultats de statique comparative pour les paramètres qui influencent les défections.

Le dernier chapitre est également consacré aux groupes d'assurance informelle. Je reviens sur les deux procédures d'autodiscipline exogènes les plus répandues dans la littérature. L'objectif est que toute procédure d'autodiscipline qui se manifeste par la rupture des liens, peut se placer entre ces deux extrêmes. Je propose une structure organisationnelle des assurances informelle, construite à la lumière des faits empiriques et je m'intéresse à leur stabilité. Dans un premier temps, je caractérise les groupes de transfert stables en n'imposant aucune restriction sur le facteur d'escompte (il faut noter que souvent dans la littérature on impose des facteurs d'escompte extrêmes, ce qui facilite les calculs). Je donne les bornes inférieure et supérieure des groupes de transfert stables émanant de réseaux interrelationnels. Je dégage ensuite les structures de réseaux qui sont le plus susceptibles d'aboutir à des groupes de transfert stables. J'identifie une série de paramètres qui contribuent à la stabilité et je produis des résultats de statique comparative sur ces paramètres. Je fais des simulations pour tenter de comparer les contributions de ces paramètres à la stabilité. A la fin je propose une procédure à pas finis (il existe une procédure dans la littérature mais à pas infinis), qui permet d'obtenir le meilleur lissage de consommation, qui est adaptée aux coalitions imbriquées.

Chapitre 1

Overlapping coalitions, bargaining and networks

1.1 Introduction

There are many situations where a group of agents can perform a task more efficiently than a single agent. A desirable behavior in this case is the formation of coalitions : “alliances among individuals or groups which differ in goals” (Gamson, 1961). When a coalition is formed, its members define rules to follow together. Therefore, forming a coalition can be viewed as writing an agreement together in order to reach some common objectives.

In politics, environmental issues, provision of public goods, customs unions, and many other situations, people write agreements. The literature on coalition formation, both applied and theoretical, is very rich. However, this literature is mostly restricted to models where each agent can be a member of no more than one coalition. That is, *coalitions cannot overlap*. Yet real life shows us the contrary. There are several situations where an agent can be signatory to an agreement to form a coalition S and at the same time, also be signatory to another agreement to form a coalition S' which differs from S. We observe such behavior in many economic, social and political situations.

In international trade, Free Trade Agreements (FTAs) involve overlapping coalitions of countries. The number of Regional Trade Agreements (RTAs) is growing rapidly all over the world. According to the World Trade Organization (WTO), it is estimated that more than half of world trade is now conducted under such agreements. Among the best known RTAs are the European Union (EU), the European Free Trade Association (EFTA), the North American Free Trade Agreement (NAFTA), the Association of Southeast Asian Nations (ASEAN) and so on. Many countries are signatories to more than one of these RTAs. Israel, for example, has an FTA with both the United States and the EU. Norway is signatory to the EFTA and has an FTA with the EU and the Baltic states.

Credit markets in developing countries are not perfect. A person with low income has limited access to credit. Rotating Saving and Credit Associations (ROSCAs) are an alternative to formal credit. A ROSCA is "an association of men and women who meet at regular intervals, for instance, once a month, and distribute a lump sum of money to one of its members" (van den Brink and Chavas, 1997). Such credit institutions are a widespread phenomenon in developing economies around the world [1]. In these economies, most people are members of multiple ROSCAs at the time (van den Brink and Chavas, 1997) : one at work, one at the family level, one at the village level and so on.

Environmental agreements also involve overlapping coalitions of countries. For example, the Asia-Pacific Partnership (APP) on clean development and climate, is an international agreement on development and transfer of technology in order to reduce greenhouse gas emissions. The signatories to APP are Australia, Canada, China, India, Japan, South Korea and the United States of America (USA). Though, unlike

1. The ROSCA is known as "tontine" in Francophone West Africa, "Dashi" among the Nupe in Nigeria, "Isusu" among the Ibo and Yoruba, and as "Susu" in Ghana. It is called "Ekub" in Ethiopia. In Tanzania, it is called "Upatu," and it is known as "Chilemba" in many other parts of East Africa. In other parts of the world, the ROSCA is called "Arisan" (Indonesia), "Pia Huey" (Thailand), "Ko" (Japan), "Ho" (Vietnam), "Kye" (Korea), and "Hui" (central China). See van den Brink and Chavas (1997) for more details.

other APP signatory countries, the USA has not ratified the Kyoto Protocol : imposition of mandatory limits on greenhouse gas emissions. But, the USA and Canada are signatories of the Convention on Long-Range Transboundary Air Pollution (LRTAP) : cutting of emissions of four pollutants (sulphur dioxide, nitrogen oxides, volatile organic compounds and ammonia) by setting country-by-country emission ceilings to be achieved by the year 2010.

Many other examples can be found in everyday economic behavior. This emphasizes the fact that agreements involving overlapping coalitions are present in a wide variety of economic and social interactions. Yet, the existing literature on coalition formation pays a little attention to overlapping coalitions. The aim of this chapter is to develop a theoretical model that will be useful for the analysis of overlapping coalitions.

As suggested Ray (2007) :

> [If the formation of coalition S leaves the worth of all other coalitions unchanged, including the worth of those groups that intersect with S, one can go ahead and simply treat each of these as separate bargaining problems. That would be the end of the story. But of course, matters are generally more complicated. The worth of a formed coalition do affect those of another, and they do so in two fundamental ways.
>
> First (law, custom, information) the formation of one coalition may negate the formation of some other coalitions [...]
>
> Second the formation of S can affect what a coalition T can achieve. Example free trade agreements within S does not preclude another for T ; but payoffs will surely be affected.]

This reflection raises some questions :

- How can we extend existing models of coalition formation to accommodate overlapping coalitions ?
- What procedure should a group of agents use to coordinate their actions ?
- What is the process that leads to the formation of overlapping coalitions ?

– How should the worth of a coalition be divided among its members ?

We provide answers to some of these questions, but more remains to be done.

The chapter is organized as follows. In Section 1.2, we mention the different steps that have been taken so far in the game-theoretic perspective on coalition formation. In Section 1.3, we build a “bargaining cover function game” that allows the formation of overlapping coalitions and we show existence of equilibria. In Section 1.4, we focus on symmetric cover function games and we provide an algorithm to compute equilibrium outcomes in this case. Finally, in Section 1.5 we establish a one-to-one link between social network structures and overlapping coalition structures.

1.2 Related literature

Myerson (1980) introduced the idea of overlapping coalitions in the economics literature. He used the terminology of “conference structure.” He was interested in allocation rules as mappings from conference structures to payoff vectors. The expression *overlapping coalition* has been well known in other disciplines like computer science and robotics [2]. The problems addressed there, however, are different from those of economic interest. Ray (2007) pointed out the lack of attention paid to overlapping coalitions in game theory. He concluded by stating the necessity of a way to model overlapping coalitions in game theory.

In the game theoretical approach to modeling coalition formation, if a coalition forms, it generally means that its members agree to behave cooperatively. But if there is more than one coalition, agents across coalitions behave non-cooperatively. This mix of cooperation and non-cooperation leads to two approaches in the literature : the blocking approach and the bargaining approach. If attention is focused on cooperation, then coalitions are treated as the fundamental behavioral units and the blocking approach is taken. On the other hand, if attention is focused on non-cooperation,

2. See Kraus et al. 1998, Hu et al. 2007, and Dang 2006 for more details.

then individuals are treated as the fundamental behavioral units and the bargaining approach is taken. Following the blocking approach, some papers have provided game theoretical models that take overlapping coalitions into account (Albizuri et al. 2006, Chalkiadakis et al. 2008, 2010). This chapter differs from these in that we follow the bargaining approach. To our knowledge, it is the first to provide a bargaining game that accommodates overlapping coalitions. In that sense, this chapter is a contribution to the literature on coalition formation and more generally on bargaining theory [3].

As this chapter follows the bargaining approach, we expose a quick overview of some models of bargaining with discounting. Rubinstein (1982) is one of the first to use the extensive-form model of bargaining. The model is of a bilateral bargaining process. But, when Binmore et al. (1985) tested bilateral bargaining of this kind on subjects, they concluded that subjects behave fairer in real life than the theory predicts. Subjects chose equal division as an obvious and acceptable compromise. Chatterjee et al. (1993) and Okada (1996) generalized the bilateral bargaining by Rubinstein (1982) to multilateral bargaining. These two papers differ in the way the proposer is selected in the bargaining process. Based on Chatterjee et al. (1993), Ray and Vohra (1999) take into account externalities across coalitions. When a coalition forms, the actions of other coalitions can affect the worth of the formed coalition. In most of these papers, once the bargaining game is introduced, an equilibrium notion is explored and the equilibrium coalition structures are characterized.

In all of these papers, however, coalitions are not allowed to overlap. Therefore, it is interesting to take a step forward by exploring the area of overlapping coalitions as suggested Ray (2007).

1.3 The Model

The model here is an extension of the one developed by Ray and Vohra (1999) to a setting where coalitions are not necessarily disjoint sets. For this purpose, we

3. Bandyopadhyay and Chatterjee (2006) is a good overview of the recent literature in this area.

introduce the *cover function (bargaining) game*. We adopt, to the extent possible, the same notations as in Ray and Vohra (1999).

1.3.1 Cover functions

Let $N \equiv \{1, 2, \ldots, n\}$ denote the set of all players. A *coalition* S is a non-empty subset of N. Two distinct coalitions S and S' are said to be *overlapping* if $S \cap S' \neq \emptyset$. A *cover* γ of N is a collection of coalitions such that

$$\gamma \equiv \{S_1, S_2, \ldots, S_m\} \text{ and } \bigcup_{k=1}^{m} S_k = N.$$

Let Γ denote the set of all covers of N. An *embedded coalition*[4] is a pair (S, γ) such that $S \in \gamma$ and $\gamma \in \Gamma$. The set of embedded coalitions is denoted by Σ and defined by $\Sigma = \{(S, \gamma) \mid S \in \gamma, \gamma \in \Gamma\}$.
A *partition* π of N is either $\{N\}$ or a cover with $S_j \cap S_k = \emptyset$ for $j \neq k$. Let Π denote the set of all partitions of N. One can easily see that $\Pi \subset \Gamma$. In the remainder, we will use the expression *overlapping coalition structure* to address an element of Γ whenever we want to emphasize on its structure. We denote for simplicity a coalition $S \equiv \{i, j, k, ...\}$ by $S = ijk...$.

Definition 1. *A cover function $v : \Sigma \rightarrow \Re$ is such that for all $(S, \gamma) \in \Sigma$, $v(S, \gamma) \geq 0$*
.

From this definition, existing notions of partition function and characteristic function can be obtained.

Definition 2.

(a) If Γ is restricted to Π, then v is a *partition function.*

(b) If in addition to (a), v is such that for all distinct pairs γ, $\gamma' \in \Pi$ and for all $S \in \gamma \cap \gamma'$, $v(S, \gamma) = v(S, \gamma')$, then v is a *characteristic function.*

4. We borrow this definition from Macho-Stadler et al. (2007).

Hence, partition functions and characteristic functions are special cases of cover functions.

1.3.2 The cover function bargaining game

For each coalition S, there is an initial *proposer* $\rho^p(S)$ in S and an order of *respondents* $\rho^r(S)$ which is a permutation of $S \setminus \rho^p(S)$. A *protocol* on N is given by the set $\rho \equiv \{(\rho^p(S), \rho^r(S))\}_{\emptyset \neq S \subseteq N}$. A *cover function (bargaining) game* is given by a triple (N, v, ρ) where N is a set of players, v is a cover function, and ρ is a protocol on N.

The game proceeds in stages. At each stage, a proposer is selected. She proposes a coalition including her and a distribution of the *conditional* coalitional worth to the other members of that coalition. Due to externalities, the actual coalitional worth is not known until the end of the game. Therefore, a proposer proposes a distribution that is conditional on the remaining coalitions that form. The respondents, the other members of the coalition to whom a *proposal* is made, either accept or reject this proposal in the order specified by the protocol. If all the respondents accept the proposal, the coalition forms. The *bargaining process* consists of a succession of proposals and responses. The *current state of the game* consists of the proposals that have already been made, the corresponding responses, the coalitions that have already formed, and the ones that can form. We provide below a formal definition of a proposal.

At each stage of the game, the set of proposers consists of the players who do not belong to any formed coalition. The set of respondents, however, is the whole set of players N. In order to permit overlapping coalitions, we allow a member of a coalition that has already formed (during the bargaining process) to continue participating in the game, but only as respondent. For example, consider a set of countries. Suppose one country takes the first step to form an RTA with some of the others. Once this RTA is formed, another country (which does not belong to the RTA that has already formed) may initiate the formation of another RTA. He can decide to extend the new proposal to some countries that belong to the already formed RTA. If the new RTA is formed, then we see overlapping RTAs.

At any stage of the game where a coalition S is proposed, let $\Gamma(S) \equiv \{\gamma \in \Gamma \mid S \in \gamma\}$ be the set of coalition structures *compatible* with S. Consider a stage of the game such that a collection of coalitions $\lambda \equiv (S_1, \ldots, S_L)$ has already formed and a coalition S is proposed. Let $\Gamma(\lambda, S)$ denote the set of coalition structures that are *compatible* with λ and S.

Formally, $\Gamma(\lambda, S) \equiv \{\gamma \in \Gamma(S) \mid \gamma \in \Gamma(S_l) \text{ for all } l = 1, \ldots, L\}$.

Definition 3. *A proposal is a pair* (S, y)*, where* $y \equiv (y(\gamma))_{\gamma \in \Gamma(\lambda,S)}$ *such that for all* $\gamma \in \Gamma(\lambda, S)$*,* $y(\gamma)$ *is a vector of size* $|S|$ *and* $\sum_{i \in S} y_i(\gamma) = v(S, \gamma)$*.*

Remark 1.

A proposal (S, y) is such that S is a non-empty subset of N and each $y(\gamma) = (y_i(\gamma))_{i \in S}$ is a distribution of $v(S, \gamma)$ among the members of coalition S conditional on the formation of γ.

In the above definition, $y_i(\gamma)$ is the payoff proposed to i contingent on the formation of the cover $\gamma \in \Gamma(\lambda, S)$.

An *outcome* of the game consists of the collection of all coalitions that form during the bargaining process if the game ends. A strategy of a player is a complete plan of action that specifies the choices to be made at each time she makes a decision. This plan of action is contingent on the history of the game. A stationary strategy of a player is to either make a proposal (possibly probabilistic) or to respond to a proposal (possibly probabilistic), conditional only on the current state of the game. A *subgame stationary perfect equilibrium* is a collection of stationary strategies such that there is no history at which a player benefits by deviating from her prescribed strategy.

We make the behavioral assumption that *if a player is indifferent between making non-acceptable proposals such that the game continues forever and making an acceptable proposal such that the game ends, she chooses to make the acceptable proposal.* The idea behind this assumption is that no individual is ill-intentioned to act as to reduce others worth without gaining any.

1.3.3 Timing of the game

At each stage $k = 1, 2, \cdots$ of the game, let P_k be the set of possible proposers. The game starts with $P_1 \equiv N$. The timing of the game is as follows.

At stage k :

(i) The protocol ρ designates $i = \rho^p(P_k)$ as the initial proposer. Player i chooses a coalition $S \subseteq N$ to which she belongs and makes a proposal to the other members of S.

(ii) The players in $S \setminus \{i\}$ respond sequentially according to the protocol $\rho^r(S)$. If all of them accept the proposal, then coalition S forms. The game moves to stage $k+1$ with $P_{k+1} \equiv P_k \setminus S$. In this case, the game continues with a new proposal made by player $\rho^p(P_{k+1})$.

(iii) In case of rejection at stage k, the game moves to stage $k+1$ but the next proposer is not designated by the protocol. Instead, she is the first rejector that belongs to P_k.[5] After a rejection, a lapse of one unit of time occurs.[6] This imposes a geometric cost on all players, and is captured by a common discount factor $\delta \in (0, 1)$. After the next proposal is made, the game continues exactly as described in (ii).

(iv) If, during the bargaining process, a player j makes a proposal that is rejected only by players in $N \setminus P_k$, then j remains the next proposer.[7]

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5. It can be the case that the first rejector, according to the protocol, is a player who already belongs to a formed coalition. In this case, the next proposer is the following rejector according to the protocol. If this player also belongs to a formed coalition, then the next proposer shifts to the following player and so forth.

6. The bargaining process can occur during a meeting. So in case of rejection, the lapse of time captures the additional renegotiation costs. The literature on discounting often explain this lapse of time as a cost of reorganizing another meeting or another discussion panel. But if the proposal is accepted, there is no additional cost and therefore there is no lapse of time.

7. Step (iv) in the timing of the game captures the fact that making an unacceptable proposal to an individual who belongs to an already formed coalition is strategically unnecessary. On an equilibrium path an unacceptable proposal is made strategically to force the rejector (who belongs

(v) The game ends if $P_{k+1} = \emptyset$. At this time an overlapping coalition structure forms. Each coalition is now required to allocate its worth among its members according to the proposal they have accepted.

Remark 2.

– If the bargaining process continues forever, all players receive zero due to discounting.

– At each stage of the game, while only members of P_k may propose, any member of N may be a respondent.

– Step (iv) is useful for the description of all possible situations. But, due to discounting, this never happens on an equilibrium path when the game ends.

Proposition 1. *An outcome of a cover function (bargaining) game is in* Γ.

Proof.

The proof is straightforward because the game ends if and only if it reaches a stage k such that $P_{k+1} = \emptyset$. □

Notice that at any stage of the game, a proposal can be made to the whole set of players, including those players who have already formed a coalition. Therefore, coalitions may overlap in an outcome.

Example 1.

Consider a small village of three households 1, 2, and 3. Suppose that in this village, 1 and 3 are enemies and 2 is a household of very skilled (or wealthy) persons. So everybody wants to sign an agreement with 2. If 1 and 3 are in the same group, their relationship will negatively affect the performance of the group. If households are allowed to bargain and form groups strategically in order to perform some tasks, what is the structure that will arise ?

To model this situation, let $N = \{1, 2, 3\}$ be the set of players. The game is given by

to P_k) to make the next proposal. It worth noting that Ray and Vohra (1999) provide an example of equilibrium with rejection in the equilibrium path.

Table 1.1.

Protocol	Covers	Cover function
$\rho^p(N) = 1$	$\gamma_1 = \{N\}$	$v(N, \gamma_1) = 4$
$\rho^p(12) = 1$	$\gamma_2 = \{1, 2, 3\}$	$v(i, \gamma_2) = 1$ for $i = 1, 2, 3$
$\rho^p(13) = 1$	$\gamma_3 = \{12, 23\}$	$v(12, \gamma_3) = v(23, \gamma_3) = 3$
$\rho^p(23) = 2$	$\gamma_4^k = \{ij, k\}$	$v(ij, \gamma_4^k) = v(k, \gamma_4^k) = 1$ (*)
$\rho^r(N) = 2 \rightarrow 3$		$v(S, \gamma) = 0$ for any other $(S, \gamma) \in \Sigma$

(*) i, j, k are such that $\{i, j, k\} = N$
$v(S, \gamma) = 0$ for any other $(S, \gamma) \in \Sigma$.

Table 1.1 – 3-person cover function game

Suppose that the distribution of coalitional worth is fixed exogenously to be equal division.[8] One possible equilibrium action sequence in the cover function bargaining game is :

(i) Player 1 makes the proposal $(12, y)$ to player 2 , with $y = (y(\gamma))_{\gamma \in \Gamma(12)}$, where

$y(\{12, 3\}) = (1/2, 1/2)$

$y(\{12, 13\}) = (0, 0)$

$y(\{12, 23\}) = (3/2, 3/2)$.

(ii) Player 2 accepts the proposal and then the coalition 12 forms.

(iii) Player 3 makes a proposal $(23, y')$ to player 2 , with $y' = y(\{12, 23\}) = (3/2, 3/2)$.

(iv) Player 2 accepts the proposal and the game ends.

It turns out that the cover γ_3 forms in equilibrium. This equilibrium coalition structure is an overlapping coalition structure. This is not possible in the partition function framework.

If we were to restrict ourselves to the partition framework, since $v(S, \gamma) \geq 0$, we would have $v(12, \gamma_3) = v(23, \gamma_3) = 0$. Obviously, the grand coalition would form in

8. Equal division is a mild requirement imposed here for simplicity as by Ray and Vohra, 1999. In real life people chose equal division (Binmore et al., 1985) as an obvious compromise. Equal division is also obtained when δ tends to unity (Ray and Vohra, 1999).

equilibrium. It is interesting that with γ_3, one can easily construct a division of the equilibrium coalitional worths, such that all the players are better off compared to all possible payoffs that they get in the partition function case. Thus, *allowing the formation of overlapping coalitions can lead to Pareto improvements*. This example shows that, apart from the descriptive aspect, a framework that accommodates overlapping coalitions is normatively interesting.

1.3.4 Existence of equilibrium

As in Ray and Vohra (1999), our notion of equilibrium allows for three kinds of mixing : the choices of a coalition to propose, a distribution of the coalitional worth, and a response. Ray and Vohra (1999) prove the theorem below for the partition function case. We show that it remains true for the cover function case. The steps of the proof remain the same. However, the generalization is far from being obvious. Furthermore, unlike the partition case, our proof is by induction on the size of the set of proposers.

Theorem 1. *There exists a stationary subgame perfect equilibrium where the only source of mixing is the choice of a coalition by each proposer.*

Proof.
Notice that for a new coalition to form during the bargaining process, it needs a new proposal to be made. This requires at least one player to be a proposer. Furthermore, there can not be a new coalition formed only by players belonging to already formed coalitions. Thus for the existence of an equilibrium, it is sufficient to focus on the set of proposers.

Without loss of generality, the proof is by induction on only the size of the set of proposers. We proceed with a sequence of four lemmas. Before that, we need some additional notation.
If a coalition S forms, let $\{-S, v|_{-S}, \rho|_{-S}\}$ denote the new game, where proposals can only be made by players in $-S \equiv N \setminus S$ with :

$$v|_{-S} \equiv \left\{ v(T,\gamma)_{T\in\gamma\setminus\{S\}} \right\}_{\gamma\in\Gamma(S)} \text{ , and}$$
$$\rho|_{-S} \equiv \left\{ (\rho^p(T'), \rho^r(T)) \right\}_{\emptyset\neq T'\subseteq N\setminus S;\ T\in\gamma\setminus\{S\},\gamma\in\Gamma(S)}.$$

Similarly define $\{-\lambda, v|_{-\lambda}, \rho|_{-\lambda}\}$ as the new game obtained after a collection of coalitions

$\lambda \equiv (S_1, \ldots, S_L)$ has formed. Let $M \equiv Max_{(S,\gamma)\in\Sigma}\{v(S,\gamma)\}$. Obviously, M exists because N is a finite set. For all $\gamma \in \Gamma$, the number of coalitions in γ is at most n (the total number of players). [9] At the beginning of the game, the number of proposers is n. However, this number decreases during the bargaining process. Thus, an individual's payoff can not exceed nM. In addition to that, due to the assumption that $v(\{i\},\gamma) \geq 0$ for all $i \in N$, the equilibrium payoff for each player lies in $X \equiv [0, nM]$.

If at stage k of the game $P_k = \{i\}$, $i \in N$ will propose a coalition T. If $T \neq \{i\}$ and she knows that at least one player will reject the proposal, i can not do better than staying alone and proposing $T = \{i\}$. [10] Otherwise, she proposes a coalition $T \neq \{i\}$ such that no player will reject this proposal and she gains at least the same as her payoff if she proposes $\{i\}$. And the game ends. Thus her best strategy is either making an acceptable proposal (if she can get more than what she gains by staying alone), or staying alone (if not).

Assume that an equilibrium exists for each cover function bargaining game with less than n proposers.

Now consider the overall game with n proposers at the beginning. Suppose that a coalition S has formed. [11] The resulting game is $\{-S, v|_{-S}, \rho|_{-S}\}$ with $n - |S|$ proposers. By assumption, an equilibrium exists for this game. Fix one equilibrium strategy for any player in this subgame. Now we are going to describe equilibrium strategies

9. The number of coalitions in γ lies between 1 (if $\gamma = \{N\}$) and n (if all the coalitions in γ are singleton sets). This is due to the fact that for a coalition to be formed, we need at least one player from the set of proposers P_k

10. Notice that according to Step (iv) in the timing of the game, if i's proposal T is such that $T \neq \{i\}$ and it is rejected, then i makes the next proposal. Since $P_k = \{i\}$, all the players in $T \setminus \{i\}$ are already members of a formed coalition. Because of the discounting following a rejection and the behavioral assumption that we made, i can not make an unacceptable proposal.

11. As far as the game is not degenerated, ie. there exists $(S,\gamma) \in \Sigma$ such that $v(S,\gamma) > 0$, at least the coalition S will form at equilibrium.

after S had formed.[12]

For this purpose, we have to generate :

(i) A probability distribution β^S over the set $\Lambda(S) \equiv \big\{\lambda = \gamma \setminus \{S\}, \gamma \in \Gamma(S)\big\}$. Each $\lambda \in \Lambda(S)$ is a collection of coalitions such that $\{S\} \cup \lambda \in \Gamma(S)$.

(ii) A set $\mathcal{U} \equiv \left\{ \big(u_l(T)\big)_{l \in T}, T \in \lambda \right\}$ of equilibrium payoff vectors (in X) for all the players in the coalitions to be formed after S has formed.

At the beginning of the game, let $i \equiv \rho^p(N)$.[13] Let $\mathcal{N}_i \equiv \{S \subseteq N, i \in S\}$ denote the set of all coalitions containing i. Let $\mathcal{A}_i \equiv \big\{\mathcal{N}_i \times \{j\}, j \in N \setminus \{i\}\big\}$, $\Delta_i \equiv \{$ Probability distribution over $\mathcal{A}_i\}$, and $\Delta \equiv \prod_{k \in N} \Delta_k$. Player $i \in N$ can make acceptable proposals to players in $S \in \mathcal{N}_i$ and unacceptable proposals to $j \in N \setminus \{i\}$.[14] A typical element $\alpha_i \in \Delta_i$ stands for i's probabilistic choice concerning the coalition to form or the other player to whom an unacceptable proposal is made. Thus, $\alpha_i(S)$ denotes the probability with which i chooses to make an acceptable proposal to $S \in \mathcal{N}_i$, and $\alpha_i\big(\{j\}\big)$ denotes the probability with which i chooses to make an unacceptable proposal to $j \in N \setminus \{i\}$. Let x_k^S denote the expected equilibrium payoff that $k \in N$ receives in the game, coming from the formation of a coalition S that she belongs to, if i is the first proposer. Notice that ex-post, if S is not in the equilibrium cover, then $x_k^S = 0$. Thus, the expected equilibrium payoffs x_k that k receives in the game, if i is the first proposer, is the sum of what she receives from each coalition that she belongs to. Formally, $x_k = \sum_{S \in \mathcal{N}_k} x_k^S$.

Fix $\alpha \in \Delta$ and $x \equiv (x_k)_{k \in N} \in X^n$. Player $i \in N$ has two options : making an acceptable proposal to a coalition $S \in \mathcal{N}_i$ or making an unacceptable proposal to $j \neq i$.

Lemma 1. *If i makes an acceptable proposal to S, then her optimal expected payoff is*

12. Notice that the protocol was suppose not to depend on the process that leads to the formation of a coalition S.

13. Player i is the first proposer of the game.

14. We can assume without loss of generally that $i \in N$ make unacceptable proposals only to singleton sets of players. The point is that for a proposal to be rejected, it needs only one player to reject it.

$g_i(S,x) \equiv \sum_{\lambda\in\Lambda(S)} \beta^S(\lambda)v(S,\gamma) - \delta\sum_{j\in S;j\neq i} x_j,$ *where* $\gamma = \{S\}\cup\lambda$.

Proof.

To alleviate the notation in the proof, we write (S,λ) instead of $\{S\}\cup\lambda$.

Player i names a coalition $S\in\mathcal{N}_i$ and makes an acceptable proposal $y(S,\gamma)$ conditional on $\gamma\in\Gamma(S)$. To do so, she solves the following program :

$$Max_y\mathbf{E}\Big\{y_i\big(S,(S,\lambda)\big)\Big\} \equiv \sum_{\lambda\in\Lambda(S)} \beta^S(\lambda)y_i\big(S,(S,\lambda)\big) \tag{1.1}$$

Subject to :

$$\sum_{\lambda\in\Lambda(S)} \beta^S(\lambda)y_j\big(S,(S,\lambda)\big) \geq \delta x_j \text{ for all } j\in S, j\neq i \tag{1.2}$$

$$\sum_{k\in S} y_k\big(S,(S,\lambda)\big) \leq v\big(S,(S,\lambda)\big) \tag{1.3}$$

(1) : $i\in N$ maximizes her payoff expecting that the cover $\gamma=(S,\lambda)$ will form with probability $\beta^S(\lambda)$.

(2) : $j\in S\setminus\{i\}$ accepts the offer. [15]

(3) : The aggregate payoff of all players in S can not be greater than the coalitional worth.

It is straightforward to see that (1.2) and (1.3) will bind at equilibrium.

Thus, at equilibrium :

$$\sum_{\lambda\in\Lambda(S)} \beta^S(\lambda)y_j\big(S,(S,\lambda)\big) = \delta x_j \text{ for all } j\in S, j\neq i \tag{1.4}$$

$$\sum_{j\in S} y_j\big(S,(S,\lambda)\big) = v\big(S,(S,\lambda)\big) \tag{1.5}$$

Sum (1.4) over j and add $\sum_{\lambda\in\Lambda(S)} \beta^S(\lambda)y_i\big(S,(S,\lambda)\big)$. Let $g_i(S,x)$ denote the maximum value of the program. From (1.5), $g_i(S,x) = \sum_{\lambda\in\Lambda(S)} \beta^S(\lambda)v\big(S,(S,\lambda)\big) - \delta\sum_{j\in S;j\neq i} x_j$.

□

15. Because she is better off than what she could gain if she rejects the proposal and becomes the next proposer.

In the previous lemma, g_i is a continuous function of x (projections and sum) and is independent of λ.

The player $i = \rho^p(N)$ will choose a coalition S in $\mathcal{N}_i$ and make an acceptable proposal to players in S if this coalition induces the highest $g_i(S,x)$ to her.

We have fixed $(x,\alpha) \in X^n \times \Delta$. For each $k \in N$, let $v_i^k(x,\alpha)$ denote i's expected payoff received if k proposes at this stage. If k's proposal is accepted, let B_i^k denote i's expected payoff. Otherwise, if k's proposal is rejected by j [16], then a new proposal is made by $\bar{j}$. Notice that $\bar{j} \equiv j$ if at this stage j is in the set of proposers, otherwise, $\bar{j} \equiv k$. [17] Thus,

$$v_i^k(x,\alpha) = B_i^k + \delta \sum_{j \neq k} \alpha_k\big(\{j\}\big) v_i^{\bar{j}}(x,\alpha). \tag{1.6}$$

Lemma 2. *For each $k \in N$, $v_i^k(x,\alpha)$ is a continuous function of (x,α).*

Proof.

For each $k \in N$, $v_i^k(x,\alpha)$ is defined by

$$v_i^k(x,\alpha) = B_i^k + \delta \sum_{j \neq k} \alpha_k\big(\{j\}\big) v_i^{\bar{j}}(x,\alpha) \tag{1.7}$$

Remind that i is the first proposer. Thus,

$$B_i^i \equiv \sum_{S \in \mathcal{N}_i} \alpha_i(S) g_i(S,x) \text{ , and for } j \neq i \text{ , } B_i^j \equiv \delta \sum_{T \in \mathcal{N}_j,\ i \in T} x_i^T \alpha_j(T) + \sum_{T \in \mathcal{N}_j,\ i \notin T} \alpha_j(T) u_i(T) \tag{1.8}$$

(1.7) : With probability $\alpha_i(S)$, i chooses a coalition S and makes an acceptable proposal to its members.

(1.8) : Player i's proposal is rejected by $j \neq i$ who becomes the next proposer and makes an acceptable proposal to T. For the first expression, i belongs to the coalition T proposed by j. For the second expression, i does not belong to the coalition T

16. Player k makes an unacceptable proposal to j with probability $\alpha_k\big(\{j\}\big)$.
17. This situation occurs according to Step (iv) in the timing of the game.

proposed by j.

For each $k \in N$, the value $v_i^k(x,\alpha)$ depends on i's best payoff g_i , and on the vector α.

The set of equations defining the value can be defined as

$$V_i \equiv (v_i^k)_{k\in N} \text{ , and } B_i \equiv (B_i^k)_{k\in N}$$

Let C denote the $n \times n$ matrix with 1's on the diagonals and $-\delta\alpha_k\big(\{j\}\big)$ as the $k\bar{j}$th off-diagonal element. Thus, $B_i = CV_i$.

In each row, the sum of the off-diagonal elements lies in $(-1;0]$ and C is nonsingular. Thus, $V_i = C^{-1}B_i$ by (1.6). We conclude that for each $k \in N$, v_i^k is continuous in x and α. Player i's expected payoff is a continuous function of x and α, whether her proposal is accepted or not. □

For each $(x,\alpha) \in X^n \times \Delta$ and a fixed $i \in N$, define $v_i(x,\alpha,\cdot)$ on Δ_i by

$$v_i(x,\alpha,\alpha_i') \equiv \sum_{S\in\mathcal{N}_i} \alpha_i'(S)g_i(S,x) + \delta \sum_{j\neq i} \alpha_i'\big(\{j\}\big)v_i^{\bar{j}}(x,\alpha)$$

and maximize this function with respect to $\alpha_i' \in \Delta_i$. Let $\phi_i^1(x,\alpha) \equiv Max_{\alpha_i'\in\Delta_i}\Big\{v_i(x,\alpha,\alpha_i')\Big\}$, $\phi_i^2(x,\alpha) \equiv Argmax_{\alpha_i'\in\Delta_i}\Big\{v_i(x,\alpha,\alpha_i')\Big\}$, and $\Phi \equiv \prod_i \phi_i^1 \prod_i \phi_i^2$.

Lemma 3. *Φ is a mapping on $X^n \times \Delta$ and it admits a fixed point $(\overline{x},\overline{\alpha})$.*

Proof.

According to the maximum theorem and the facts that $v_i(x,\alpha,\alpha_i')$ is continuous, and $\phi_i^1(x,\alpha)$ is a continuous function, $\phi_i^2(x,\alpha)$ is a convex-valued upper hemicontinuous correspondence. This result holds for all i in N and (x,α) in $X^n \times \Delta$ because i and (x,α) where chosen arbitrarily. Thus, $\prod_i \phi_i^1$ maps $X^n \times \Delta$ on X^n. Therefore, the correspondence $\Phi : X^n \times \Delta \to X^n \times \Delta$ admits a fixed point $(\overline{x},\overline{\alpha})$ according to Kakutani's fixed point theorem. □

Lemma 4. *The fixed point of Φ induces an equilibrium.*

Proof.

Let σ denote the strategy profile such that at the stage k of the game :

(i) When $P_k = N$, an arbitrary $i \in N$, as a proposer, makes proposals according to $\overline{\alpha}$:

- To every coalition $S \in \mathcal{N}_i$ such that $\overline{\alpha}_i(S) > 0$, she proposes $y(S, \gamma)$ which solves the maximization problem addressed in the equations (1) – (3).
- To every $j \neq i$ such that $\overline{\alpha}_i\big(\{j\}\big) > 0$, she offers, for every possible cover containing the coalition $\{i, j\}$, less than $\delta \overline{x}_j^{\{i,j\}}$.

(ii) Suppose that $P_k = N$, $i \in N$ is a respondent to a proposal $y(S, \gamma)$, and every respondent j to follow i is offered an expected payoff of at least $\delta \overline{x}_j$, i.e. $\sum_{\lambda \in \Lambda(S)} \beta^S(\lambda) y_j\big(S, (S, \lambda)\big) \geq \delta x_j$ for all respondents j that follow i. Then i accepts the proposal if and only if $\sum_{\lambda \in \Lambda(S)} \beta^S(\lambda) y_i\big(S, (S, \lambda)\big) \geq \delta x_i$.

(iii) Suppose that $P_k = N$, and $i \in N$ is the first respondent. From (ii), if there is exactly one respondent to follow i, say $j \in N$, such that j is offered an expected value less than δx_j, then j will reject the proposal. Player i's decision will now depend on the present value of the payoff resulting from j rejecting the offer and this will lead to a new proposal as in (i). This value is precisely $\delta v_i^{\overline{j}}(\overline{x}, \overline{\alpha})$. Player i accepts the proposal if and only if $\delta v_i^{\overline{j}}(\overline{x}, \overline{\alpha}) \geq \delta x_i$. Notice that this inequality might hold even though we know from the construction of v_i^j and the fact that $(\overline{x}, \overline{\alpha})$ is a fixed point, that $\delta v_i^j(\overline{x}, \overline{\alpha}) \leq \overline{x}_i$.

Now consider a proposal made to respondents $\{1, ..., r\}$ in that order. Suppose we have computed the decisions of all respondents $i+1, ..., r$. Player i's decision is then obtained by considering the decision of the next responder to reject the proposal, say j. Player i accepts the proposal if and only if $\delta v_i^{\overline{j}}(\overline{x}, \overline{\alpha}) \geq \delta x_i$. In this way we obtain a complete description of the actions of all respondents of a proposal.

(iv) Suppose that $P_k \subset N$. It must be the case that some collection of coalitions $\lambda = (S_1, \ldots, S_L)$ has already formed. The strategies of the remaining players are defined according to the preselected equilibrium of the game $\{-\lambda, v|_{-\lambda}, \rho|_{-\lambda}\}$.

We show that a strategy profile σ satisfying (i)-(iv) is a stationary equilibrium.
Consider σ and deviations that a single $i \in N$ can contemplate.
By construction $\overline{x}_i = v_i(\overline{x}, \overline{\alpha}, \overline{\alpha}_i) = Max\Big\{v_i(\overline{x}, \overline{\alpha}, \cdot)\Big\}$. Thus, it is not possible for i, as a proposer, to receive a higher payoff than $\overline{x}_i$ by making a one-shot deviation from $\overline{\alpha}_i$.
This implies that no other strategy can yield i a higher payoff than $\overline{x}_i$. The action prescribed in (i) achieves $\overline{x}_i$ and therefore cannot be improved upon. Suppose that i is a respondent and all respondents to follow i are offered at least $\delta\overline{x}_j$, which, by hypothesis, they will accept. By deviating, i gets a present value of $\delta\overline{x}_i$. Clearly then, the action prescribed in (ii) cannot be improved upon. Suppose i is a respondent who is followed by a respondent j who, based on σ, will reject the proposal. Accepting the proposal yields $\delta v_i^{\overline{j}}(\overline{x}, \overline{\alpha})$ to $i \in N$ while rejecting it yields at most $\delta\overline{x}_i$. Thus, the action described in (iii) cannot be improved upon. A similar argument applies to the description in (iii) of i's actions in the other cases when i is a responder. Finally, notice that when some players have left the game, i can not do better than the actions in (iv). Thus, σ is a stationary equilibrium. □

This Lemma ends the induction. The strategy profile σ is then a stationary subgame perfect equilibrium where the only source of mixing is in the choice of a coalition by each proposer. □

Remark 3.

The steps of this proof are exactly the same as in the partition function case (Ray and Vohra 1999). Nevertheless, there are two fundamental differences. First, unlike the partition case, the induction is on the number of potential proposers during the bargaining process. In the cover case, at every stage of the game, the number of possible responders is fixed at n. This is quite different from what happens in the partition case. Secondly, unlike the partition case, a player can be a responder to acceptable proposals more than once. These features have been taken into account in this proof through the new element x_j^S denoting the maximum payoff that $j \in N$ can obtain from the formation of the coalition S if $i \in N$ is the first proposer in the game.

1.4 Symmetric cover function games

In this section, we provide an algorithm to compute an equilibrium coalition structure for *symmetric cover function games.*

Symmetric partition function (Ray and Vohra, 1999)

Let N be a set of n players, and π a partition of N. If $\pi = \{S_1, S_2, \ldots, S_m\}$ with $|S_i| = s_i$, then the *numerical coalition structure* of π is $\mathbf{n}(\pi) \equiv (s_1, s_2, \ldots, s_m)$. Notice that $\sum_{i=1}^{m} s_i = n$. A *partition function* v is *symmetric* if for a given partition π and a coalition $S_i \in \pi$, $v(S_i, \pi) = v(s_i, \mathbf{n}(\pi))$. Thus, the numerical coalition structure represents the equivalence class of symmetric coalitions (in consideration of the symmetric partition function).

As the coalition formation game has both cooperation (within coalitions) and non-cooperation (across coalition) aspects, the definition of a symmetric cover function game is not straightforward. A player may belong to more than one coalitions. How will this player behave ? Will she behave non-cooperatively against the other members of a coalition that she belongs to ? The answer is not trivial. That is one of the reasons why we favor the bargaining approach. As side payments are allowed, one can think that being a member of multiple coalitions will have an effect on the bargaining power of these coalitions and that will affect in either direction (reduce or increase) the worth of the concerned coalitions. This can induce a specific effect on the division of the worth in those particular coalitions. We do not address this problem here but one should keep it in mind.

1.4.1 Symmetric cover function

Let $\gamma \equiv \{S_1, \ldots, S_m\} \in \Gamma$. Let $\mathbf{n}(\gamma) \equiv (s_1, ..., s_m)$ denote the numerical coalition structure of γ. Contrary to the partition case, $\sum_{i=1}^{m} s_i \geq n$ and if there exists at least one overlapping player,[18] $\sum_{i=1}^{m} s_i > n$. Thus, the numerical structure used in the partition case is not sufficient to characterize an overlapping coalition structure.

For each $S_i \in \gamma$, count the number of overlapping players and identify the number of distinct coalitions that these players belong to. Formally, let $\mathbf{o}(S_i) \equiv \{(o_1, c_1), \ldots, (o_l, c_l)\}$ denote a set of pairs of integers. For each pair, the first integer accounts for the number of overlapping players, and the second one for the number of distinct coalitions that these overlapping players belong to. The set $\mathbf{o}(S_i)$ stands for, o_1 players belong to c_1 coalitions, o_2 players belong to c_2 coalitions,..., o_l players belong to c_l coalitions. For further simplicity, we will rank these pairs by using the lexicographical order. This collection of pairs is denoted *the overlapping status* of S_i. Similarly, define the overlapping status of any collection of coalitions λ that we will denote by abuse of notation $\mathbf{o}(\lambda)$.

Once $\mathbf{n}(\lambda) \equiv (s_1, ..., s_m)$ and $\mathbf{o}(\lambda) \equiv \{(o_1, c_1), \ldots, (o_l, c_l)\}$ are defined, let $K(\lambda) \equiv \sum_{i=1}^{m} s_i - \sum_{j=1}^{l} (c_j - 1) o_j$.

Proposition 2.

For all $\gamma \in \Gamma$, $K(\gamma) = n$.

Proof.

The proof is straightforward. Notice that we may have multiple counts of players in the sum of the sizes of coalitions in γ (because some players may belong to more than one coalition). Therefore, to obtain the exact number of players, we need to subtract this multiple counts so that each player is counted only once. □

18. The set γ is not necessarily a partition.

Example 2. Let $N = \{1, 2, 3, 4, 5\}$. In Table 1.2 we verify Proposition 2 for some covers.

γ	$\mathbf{n}(\gamma)$	$\mathbf{o}(\gamma)$	$\sum s_i$	$\sum(c_j - 1)o_j$	$K(\gamma)$
$\{12345\}$	(5)	$\{(0,0)\}$	5	0	5
$\{123, 345\}$	$(3,3)$	$\{(1,2)\}$	6	1	5
$\{123, 134, 345\}$	$(3,3,3)$	$\{(1,3);(2,2)\}$	9	4	5
$\{123, 124, 125\}$	$(3,3,3)$	$\{(2,3)\}$	9	4	5

TABLE 1.2 – Number of distinct players in a cover

Definition 4. *Let $\gamma \equiv \{S_1, \ldots, S_m\} \in \Gamma$ and $s_i = |S_i|$ for all $S_i \in \gamma$. The representative coalition structure of γ is denoted $\boldsymbol{r}(\gamma)$ and defined by $\boldsymbol{r}(\gamma) \equiv \big\{\{s_i; \boldsymbol{o}(S_i)\}, i = 1, 2, \ldots, m\big\}$.*

A cover function is *symmetric* if the worth of a coalition in a given cover depends *only* on the size and the overlapping status of each coalition in that cover.

Definition 5. *A cover function v is symmetric if for each embedded coalitions $(S, \gamma) \in \Sigma$, $v(S_i, \gamma) = v\big(\{s_i; \boldsymbol{o}(S_i)\}, \boldsymbol{r}(\gamma)\big)$.*

Remark 4.

- By definition, the representative coalition structure represents the equivalence class of symmetric coalitions (in consideration of the symmetric cover function). Therefore, for all $S_i \in \gamma$, $v(S_i, \gamma)$ can be written as $v\big(\{s_i; \mathbf{o}(S_i)\}, \mathbf{r}(\gamma)\big)$. [19]
- If γ is a partition, then for all $S_i \in \gamma$, $\mathbf{o}(S_i) = \big\{(0; 0)\big\}$. In this case the representative coalition structure is isomorphic to the numerical structure. Thus, the numerical coalition structure is a particular case of representative coalition structures.

19. We need this notation for further use in the algorithm.

To illustrate the symmetric cover function, let $\gamma = \{12, 34, 45\} \in \Gamma$. Let v be a symmetric cover function on γ. Thus, $v(34, \gamma) = v(45, \gamma)$. The equality does not necessary hold for the coalition 12 even if it is also of size 2, because $\mathbf{o}(12) \neq \mathbf{o}(34)$. A *symmetric* cover function (bargaining) game is a cover function (bargaining) game such that the value function is a symmetric cover function.

1.4.2 The algorithm

In this section we extend the algorithm by Ray and Vohra (1999) to symmetric cover functions. Even if the steps are alike, there are three fundamental differences. First, the representation of the equivalence class of symmetric coalitions is completely different. Secondly, the induction is made on the number of *distinct* players. Thirdly, there is a difference between the optimal size "t" of a new coalition to form, and the number "t'" of "new" players in this coalition. A new player is one who does not belong to an already formed coalition. The algorithm is based on the backward induction and computes at each stage the optimal size and overlapping status of the next coalition to form.

Consider an n-person symmetric cover function (bargaining) game. Let $\gamma \in \Gamma$ and let $\lambda \subseteq \gamma$ denote a collection of coalitions. The sets $\mathbf{o}(\cdot)$, $\mathbf{r}(\lambda)$, and the integer $K(\lambda)$ are defined as in the previous section. Notice that $K(\emptyset) = 0$. By abuse of notation, we will use the previous notations without their arguments (in case of no confusion) . Let $\mathcal{F} \equiv \{\mathbf{r} \mid \ K(\mathbf{r}) < n\}$. In the algorithm we construct a mapping $t : \mathcal{F} \to \mathbb{N}^*$. This $t(\cdot)$ will characterize the optimal size of the next coalition to form and it will induce an integer $t' \leq t$ that will characterize the number of new players in this coalition. Remember that at the stage k of the game, only a player in P_k can make a new proposal. Starting from $\emptyset$, we apply $t(\cdot)$ repeatedly and we can generate a particular set $\mathbf{r}^*$.

Step 1 Let $\mathbf{r} \in \mathcal{F}$ such that $K(\mathbf{r}) = n - 1$. Let $t(\mathbf{r})$ denote the largest [20] integer

20. In the case where we have more than one coalition, choose the largest coalition that achieves the desired outcome. One can also choose the one in which there is no overlapping player or in

in $\{1,\ldots,n\}$ that maximizes the expression $\frac{v\left(\{t;\mathbf{o}\},\mathbf{r}\bullet\{t;\mathbf{o}\}\right)}{t}$ where $\mathbf{r}\bullet\{t;\mathbf{o}\}$ is the representative coalition structure obtained by concatenating $\mathbf{r}$ with $\{t;\mathbf{o}\}$. Obviously, we have $t'(\mathbf{r})\equiv 1.$[21] Let $c(\mathbf{r})\equiv\mathbf{r}\bullet\{t;\mathbf{o}\}$. Obviously, $\mathrm{K}\big(\mathrm{c}(\mathbf{r})\big)$=n.

Step 2 Recursively, suppose that we have defined $t(\mathbf{r})$ and $t'(\mathbf{r})$ for all $\mathbf{r}\in\mathcal{F}$ such that

$K(\mathbf{r})=j+1,\ldots,n-1$, for some $j\geq 0$. Suppose moreover that $K(\mathbf{r})+t'(\mathbf{r})\leq n$ for any such $\mathbf{r}$. Suppose that we have defined recursively the sets $c(\mathbf{r})$ by successive concatenations such that $K\big(c(\mathbf{r})\big)=n$.

Step 3 For any $\mathbf{r}$ such that $K(\mathbf{r})=j$, let $t(\mathbf{r})$ denote the largest[22] integer in $\{1,\ldots,n\}$ that maximizes the expression $\frac{v\left(\{t;\mathbf{o}\},c\left(\mathbf{r}\bullet\{t;\mathbf{o}\}\right)\right)}{t}$.

Step 4 For any such $\mathbf{r}$, let $t'(\mathbf{r})\equiv K\big(c(\mathbf{r}\bullet\{t;\mathbf{o}\})\big)-K\big(c(\mathbf{r})\big)$.

Step 5 Complete this recursive definition so that t and t' are now defined on all $\mathbf{r}\in\mathcal{F}$. Define a representative coalition structure of the entire set of players N by $\mathbf{r}^*$, such that $\mathbf{r}^*\equiv c(\emptyset)$.

Example 3. Let $N=\{1,2,3\}$. The following table, Table 1.3, summarizes a symmetric cover function where $v(\mathbf{r})$ denotes the collection of the values $v\big(\{t;\mathbf{o}\},\mathbf{r}\big)$.

Similarly to the previous example, one can easily see that $r^*(\gamma)\equiv\Big\{\{2;(1,2)\},\{2;(1,2)\}\Big\}$ is the equilibrium representative coalition structure. The first person to propose will chose one player and make an acceptable proposal to this player. The remaining player will make an acceptable proposal to one player in the previously formed coalition. For simplicity, we use Table 1.4 to compute the predictions of the algorithm for this symmetric cover function game.

contrary the one in which there are more overlapping players. It depends on the context. What is important here is to have a rule to choose one t in order to compute a unique equilibrium.

21. With K($\mathbf{r}$)=n-1, P_k=1.
22. Depending on the context.

γ	$\mathbf{r}(\gamma)$	$v(\mathbf{r})$
$\{123\}$	$\{3;(0,0)\}$	4
$\{1,2,3\}$	$\big\{\{1;(0,0)\},\{1;(0,0)\},\{1;(0,0)\}\big\}$	$(1,1,1)$
$\{12,23\}$	$\big\{\{2;(1,2)\},\{2;(1,2)\}\big\}$	$(3,3)$
$\{ij,k\}$ (*)	$\big\{\{2;(0,0)\},\{1;(0,0)\}\big\}$	$(1,1)$
$\{23,13\}$	$\big\{\{2;(1,2)\},\{2;(1,2)\}\big\}$	$(3,3)$
$\{12,13\}$	$\big\{\{2;(1,2)\},\{2;(1,2)\}\big\}$	$(3,3)$
		0 for any other $\mathbf{r}$

(*) i,j,k are such that $\{i,j,k\}=N$.

TABLE 1.3 – A symmetric cover function

$K(\mathbf{r})$	$\mathbf{r}$	$c\big(\mathbf{r}\bullet\{t;\mathbf{o}\}\big)$	$\frac{v\left(\{t;\mathbf{o}\},\mathbf{r}\right)}{t}$	t	t'
2	$\big\{\{1;(0,0)\},\{1;(0,0)\}\big\}$	$\big\{\{1;(0,0)\},\{1;(0,0)\}\big\}\bullet\{1;(0,0)\}$	1		
2	$\big\{\{1;(0,0)\},\{1;(0,0)\}\big\}$	$\big\{\{1;(0,0)\},\{1;(0,0)\}\big\}\bullet\{2;(1,2)\}$	0		
2	$\big\{\{2;(0,0)\}\big\}$	$\big\{\{2;(0,0)\}\big\}\bullet\{1;(0,0)\}$	1		
2	$\big\{\{2;(0,0)\}\big\}$	$\big\{\{2;(0,0)\}\big\}\bullet\{2;(1,2)\}$	**1.5**	**2**	**1**
1	$\big\{\{1;(0,0)\}\big\}$	$\big\{\{1;(0,0)\}\big\}\bullet\{2;(0,0)\}$	**0.5**	**2**	**2**
0	$\emptyset$	$\big\{\{1;(0,0)\}\big\}\bullet\{2;(0,0)\}$	1		
0	$\emptyset$	$\big\{\{2;(0,0)\}\big\}\bullet\{2;(1,2)\}$	**1.5**	**2**	**2**
0	$\emptyset$	$\big\{\{3;(0,0)\}\big\}$	1.33		
$\mathbf{r}^*=\big\{\{2;(1,2)\},\{2;(1,2)\}\big\}$					

TABLE 1.4 – The algorithm

In Table 1.4, we implement the algorithm (backward induction) for Example 3 below. According to the first series of rows, if $|P_k|=1$, then the optimal coalition structure is formed by one coalition of size 2 and the following optimal coalition to form is of size 2 and contains 1 overlapping player.

According to the second series of rows, if $|P_k|=1$, then the optimal coalition structure is formed by one coalition of size 1 and the following optimal coalition to form is of size 2 and contains no overlapping player.

Now consider the overall game. According to the third series of rows, the first optimal

coalition to form is of size 2 and contains no overlapping player. Furthermore, from the first series of row we obtain that the following optimal coalition to form is of size 2 and contains 1 overlapping player.
Thus, the algorithm yields the optimal representative coalition structure which is $\mathbf{r}^* = \big\{\{2;(1,2)\},\{2;(1,2)\}\big\}$ as predicted.

At this point, one can ask if the prediction of the algorithm is specific to the example. Does the algorithm apply for every symmetric cover function game ?
The justification to this algorithm comes from Theorem 3.1 in Ray and Vohra (1999). This theorem can also be extended to the cover case.

In the bargaining process, if a collection of coalitions λ have formed, denote $\mathbf{r}(\lambda)$ the corresponding representative coalition structure and define $a(\mathbf{r}(\lambda))$ as

$$a(\mathbf{r}(\lambda)) \equiv \frac{v\big(\{t\big(\mathbf{r}(\lambda)\big);\mathbf{o}\},\mathbf{r}(\gamma)\big)}{t}, \text{ with } \gamma \equiv c(\mathbf{r} \bullet \{t(\mathbf{r}(\lambda));\mathbf{o}\}).$$

Regularity condition :
For all $\mathbf{r}$ such that $K(\mathbf{r}) < n-1$, there exists a couple $\{s;\mathbf{o}\}$ (where s is an integer, and $\mathbf{o}$ a collection of couples of integers) such that $K\big(\mathbf{r} \bullet \{s;\mathbf{o}\}\big) = n - K(\mathbf{r})$ and $v\big(\{s;\mathbf{o}\},\mathbf{r}\big(\mathbf{r} \bullet \{s;\mathbf{o}\} \bullet \mathbf{r'}\big)\big) > 0$ for all $\mathbf{r}'$ such that $K\big(\mathbf{r}\big(\mathbf{r} \bullet \{s;\mathbf{o}\} \bullet \mathbf{r'}\big)\big) = n$.
This condition implies that for all $\mathbf{r}$ such that $K(\mathbf{r}) < n-1$, $a(\mathbf{r}) > 0$.

Theorem 2. *Under the regularity condition, there exists $\delta^* \in (0,1)$ such that for all $\delta \in (\delta^*,1)$, any equilibrium in which an acceptable proposal is made with positive probability at any stage must be of the following form. At a stage in which a collection of coalitions λ has formed, and $\boldsymbol{r} \equiv \boldsymbol{r}(\lambda)$ belongs to $\mathcal{F}$, the next coalition that forms is of size $t(\boldsymbol{r})$, its overlapping status is $\boldsymbol{o}$, and the payoff to a proposer is*

$$a(\boldsymbol{r},\delta) = \frac{v\big(\{t\big(\boldsymbol{r}(\lambda)\big);\boldsymbol{o}\},\boldsymbol{r}(\gamma)\big)}{1+\delta\big(t(\boldsymbol{r})-1\big)}, \ \gamma \equiv c\big(\boldsymbol{r} \bullet \{t(\boldsymbol{r}(\lambda));\boldsymbol{o}\}\big)$$

The representative coalition structure corresponding to any such equilibrium is $\boldsymbol{r}^*$.

Proof.

The following lemma establishes the existence of δ^*.

Lemma 5. *There exists* $\delta^* \in (0,1)$ *such that for all* $\delta \in (\delta^*, 1)$ *, and all* $\boldsymbol{r} \in \mathcal{F}$*, there is a unique integer* $t(\boldsymbol{r})$ *in the set* $\{1, 2, \ldots, n\}$*, a unique overlapping status* $\boldsymbol{o}$ *that maximizes* $\frac{v\left(\{t;\boldsymbol{o}\},c\left(\boldsymbol{r}\bullet\{t;\boldsymbol{o}\}\right)\right)}{1+\delta(t-1)}$ *, and a unique corresponding* $t' \in \{1, 2, \ldots, n - K(\boldsymbol{r})\}$*.*

Proof.

At the stage k of the game, if $P_k = 1$, then $K(\mathbf{r}) = n - 1$ and trivially, $t'(\mathbf{r}) = 1$. This means that only one remaining player will make a proposal. She can decide to be on her own or to form the largest coalition that provides him the best average payoff. Therefore, the couple $\{t; \mathbf{o}\}$, where $t(\mathbf{r})$ is the size of the next coalition to form and $\mathbf{o}$ its overlapping status, is the unique maximizer of $\frac{v\left(\{t;\mathbf{o}\},c\left(\mathbf{r}\bullet\{t;\mathbf{o}\}\right)\right)}{t}$.

We prove later that this couple is also the only one maximizer of $\frac{v\left(\{t;\mathbf{o}\},c\left(\mathbf{r}\bullet\{t;\mathbf{o}\}\right)\right)}{1+\delta(t-1)}$. (*)

Fix some $\mathbf{r} \in \mathcal{F}$ such that $K(\mathbf{r}) < n - 1$ and consider the sequence $\{\delta^q\}$ in $(0,1)$ such that $\delta^q \to 1$. Let $\mu(\mathbf{r}, \delta^q) \equiv Argmax_{\{t;\mathbf{o}\}} \left\{ \frac{v\left(\{t;\mathbf{o}\},c\left(\mathbf{r}\bullet\{t;\mathbf{o}\}\right)\right)}{1+\delta^q(t-1)} \right\}$. By the maximum theorem, this correspondence is upper hemicontinuous. Since the set of maximizers is finite, there exists δ^n such that $\mu(\mathbf{r}, \delta^q) \subseteq \mu(\mathbf{r}, 1)$ for all $\delta^q \geq \delta^n$.

Since $\mathcal{F}$ is finite, there exists δ^* such that for all $\mathbf{r} \in \mathcal{F}$, $\mu(\mathbf{r}, \delta^q) \subseteq \mu(\mathbf{r}, 1)$ for all $\delta^q \geq \delta^*$. And $\mu(\mathbf{r}, 1) = Argmax_{\{t;\mathbf{o}\}} \left\{ \frac{v\left(\{t;\mathbf{o}\},c\left(\mathbf{r}\bullet\{t;\mathbf{o}\}\right)\right)}{t} \right\}$. This means that if $\delta \geq \delta^*$, then for all $\mathbf{r}$, if $\{t; \mathbf{o}\}^*$ maximizes $\frac{v\left(\{t;\mathbf{o}\},c\left(\mathbf{r}\bullet\{t;\mathbf{o}\}\right)\right)}{1+\delta(t-1)}$, then $\{t; \mathbf{o}\}^* \in \mu(\mathbf{r}, \delta)$, and $\frac{v\left(\{t;\mathbf{o}\}^*,c\left(\mathbf{r}\bullet\{t;\mathbf{o}\}^*\right)\right)}{t^*} = \frac{v\left(\{t(\mathbf{r});\mathbf{o}\},c\left(\mathbf{r}\bullet\{t(\mathbf{r});\mathbf{o}\}\right)\right)}{t(\mathbf{r})} \equiv a(\mathbf{r})$.

It remains to show that if $\delta \geq \delta^*$, then $\mu(\mathbf{r}, \delta)$, contains only one such $\{t; \mathbf{o}\}^*$, and that t^* is the largest possible, that maximizes the average worth. Thus, $\mu(\mathbf{r}, \delta) = \{t(\mathbf{r}); \mathbf{o}\}$.

Obviously, for $K(\mathbf{r}) = n - 1$, $\{t(\mathbf{r}); \mathbf{o}\}$ is unique and $\emptyset \neq \mu(\mathbf{r}, \delta) \subseteq \mu(\mathbf{r}, 1)$. Thus,(*) is straightforward.

For $K(\mathbf{r}) < n - 1$, suppose by contradiction that this is not the case. Then there exists $\delta \geq \delta^*$ and $\{t; \mathbf{o}\}^*$ such that $t^* < t(\mathbf{r})$. For $t^* < t(\mathbf{r})$, $\frac{1-\delta}{t(\mathbf{r})} + \delta < \frac{1-\delta}{t^*} + \delta$, and according to the regularity condition ($a(\mathbf{r}) > 0$), we obtain $\frac{t(\mathbf{r})a(\mathbf{r})}{1+\delta\left(t(\mathbf{r})-1\right)} > \frac{t^* a(\mathbf{r})}{1+\delta(t^*-1)}$.

Hence $\frac{v\left(\{t;\mathbf{o}\},c\left(\mathbf{r}\bullet\{t;\mathbf{o}\}\right)\right)}{1+\delta(t-1)} > \frac{v\left(\{t;\mathbf{o}\}^*,c\left(\mathbf{r}\bullet\{t;\mathbf{o}\}^*\right)\right)}{1+\delta(t^*-1)}$. This is a contradiction because $\{t; \mathbf{o}\}^*$ is a maximizer of $\frac{v\left(\{t;\mathbf{o}\},c\left(\mathbf{r}\bullet\{t;\mathbf{o}\}\right)\right)}{1+\delta(t-1)}$. Once we have $\{t;\mathbf{o}\}$, then obviously by definition, t' is uniquely defined by $t'(\mathbf{r}) = K\big(c\big(\mathbf{r}\bullet\{t;\mathbf{o}\}\big)\big) - K\big(c(\mathbf{r})\big)$. □

Lemma 6. *Consider the stage k of the game such that a collection of coalitions λ has formed. Let $(x_l)_{l\in\mathcal{A}}$ denote a collection of equilibrium expected payoffs as defined in the general cover function game. Suppose that for each $\{t;\boldsymbol{o}\}$, the representative coalition structure following $\boldsymbol{r}\bullet\{t;\boldsymbol{o}\}$ is given by $c\big(\boldsymbol{r}\bullet\{t;\boldsymbol{o}\}\big)$.*
Then if $i\in P_k$ makes an acceptable proposal to coalition S with a positive probability :

(i) $(j\in S, j\neq i$ *and* $x_l < x_j) \Longrightarrow l\in S$

(ii) $x_i \leq x_l$ *for all* $l\in P_k$

Proof.

Player i makes an acceptable proposal to S of size s and overlapping status $\mathbf{o}(S)$. Let $c\big(\mathbf{r}\bullet\{s;\mathbf{o}(S)\}\big)$ denote the resulting representative coalition structure. Then, $x_i \geq v\big(\{s;\mathbf{o}(S)\},c\big(\mathbf{r}\bullet\{s;\mathbf{o}(S)\}\big)\big) - \delta\sum_{j\in S,j\neq i}x_j$, and this is not less than $Max_{T\in\Gamma(\lambda),i\in T}\left\{v\big(\{|T|;\mathbf{o}(T)\},c\big(\mathbf{r}\bullet\{|T|;\mathbf{o}(T)\}\big)\big) - \delta\sum_{j\in T,j\neq i}x_j\right\}$. This result holds because S is the best offer that i can make at this stage.

Then $l\notin S$ implies that $x_l \geq x_j$ because i can propose a set $T\ni l$ such that T has the same size and the same overlapping status as S. Thus, l can gain at least x_j. Therefore (i) obtains.

Suppose (ii) false, then $x_i > x_l$ for some l in P_k .

If $l\notin S$ then l (or any other player player in P_k and not in S) can form a coalition of the same size as S and the same overlapping status, by replacing player i by herself.[23] By doing so, she will receive the same payoff x_i because of the symmetric cover function. This is a contradiction.

Now suppose that $l\in S$. Thus,

$x_l \geq v\big(\{s;\mathbf{o}(S)\},c\big(\mathbf{r}\bullet\{s;\mathbf{o}(S)\}\big)\big) - \delta\sum_{j\in S,j\neq l}x_j = v\big(\{s;\mathbf{o}(S)\},c\big(\mathbf{r}\bullet\{s;\mathbf{o}(S)\}\big)\big) - \delta\sum_{j\in S,j\neq i}x_j + \delta x_l - \delta x_i$.

23. Remember that in the symmetric case, what matters for the coalitional worth if the cover is given, is the size of a coalition and its overlapping status.

Then using the previous inequality

$v\big(\{s;\mathbf{o}(S)\},c\big(\mathbf{r}\bullet\{s;\mathbf{o}(S)\}\big)\big)-\delta\sum_{j\in S,j\neq i}x_j\geq$

$Max_{T\in\Gamma(\lambda),i\in T}\Big\{v\big(\{|T|;\mathbf{o}(T)\},c\big(\mathbf{r}\bullet\{|T|;\mathbf{o}(T)\}\big)\big)-\delta\sum_{j\in T,j\neq i}x_j\Big\},$

we have

$x_l\geq x_i$ and then (ii). □

Remaining proof of the theorem.

Fix an equilibrium as described in the theorem and let δ lies in $(\delta^*,1)$ with δ^* defined as in the Lemma 5. We proceed by induction on the size of P_k, following the departure of a collection of coalitions λ. Consider a stage k of the game.

If $P_k=\{i\}$, then there is nothing to prove because i will make an acceptable proposal to the players that insure him the highest payoff.

Suppose by induction that the theorem holds at any stage where $K\big(\mathbf{r}(\lambda)\big)=m+1,\dots,n-1$ for some $m\geq 0$.

Consider now a stage where $k\big(\mathbf{r}(\lambda)\big)=m$. Let $(x_l)_{l\in\mathcal{A}}$ denote a collection of equilibrium expected payoffs as defined in the general cover function game. We will prove that if S is the coalition to form at this stage and $\mathbf{o}(S)$ its overlapping status, then $s=t\big(\mathbf{r}(\lambda)\big)$ where $s=|S|$. Since each player in P_k makes an acceptable proposal to some coalition with positive probability, it comes from induction hypothesis and (ii) of the lemma 6 that $x_i=x_j=x$ for all $i,j\in P_k$.

It follows from the induction and the optimality of the proposal that

$x=v\big(\{s;\mathbf{o}(S)\},c\big(\mathbf{r}(\lambda)\bullet\{s;\mathbf{o}(S)\}\big)\big)-\delta(s-1)x\geq v\big(\{|T|;\mathbf{o}(T)\},c\big(\mathbf{r}(\lambda)\bullet\{|T|;\mathbf{o}(T)\}\big)\big)-\delta(|T|-1)x$

for all $T\ni i$. This implies that $\{s;\mathbf{o}(S)\}\in\mu(\mathbf{r},\delta)$. By the Lemma 5 we conclude that $s=t\big(\mathbf{r}(\lambda)\big)$.

Of course the payoff to a proposer is $a(\mathbf{r},\delta)$. □

Thus, the algorithm generates an equilibrium representative coalition structure for a symmetric cover function game.

1.5 Overlapping coalition structures and networks

Networks are widely studied in social sciences and received recently considerable attention in economics. Many economic situations fit the network framework : [24] information about job opportunities (Calvo-Armengol and Jackson , 2004), trade of goods in non-centralized markets (Wang and Watts, 2006), provision of mutual insurance in developing countries (Fafchamps and Lund, 2003), research and development collusive alliance among corporations (De Weerdt, 2002), international alliances trading agreements (Furusawa and Konishi, 2007) are some among others.

A network is a collection of pairs of linked agents. The sets of directly linked agents may overlap in all the described economic situations above. In this section, we point out the link between undirected networks and overlapping coalition structures. More importantly, this link is in both directions. The intuition is the following :

- First, in a given population, an undirected network of individuals can be viewed as a collection of coalitions. Each coalition represents the collection of directly linked individuals. Since some of these coalitions may overlap, one can view the resulting structure of the population as a cover.
- Secondly, for a coalition to form, it needs the consent of all the coalitional members. Therefore, each pair of individual in a coalition is linked by this agreement. As this holds for each coalition in a cover, then the collection of such pairs induces an undirected network.

In the following we recall the basic settings of network formation theory.

24. For a survey on network formation, see Jackson (2003).

1.5.1 Networks

Let $N = \{1, 2, \ldots, n\}$ denote a set of players. A network g is a list of unordered pairs of players $\{i, j\}$, ij for simplicity. Let g^N denote the set of all subsets of N of size 2. Let $G \equiv \{g \mid \; g \subseteq g^N\}$ denote the set of all possible networks on N. Let $N(g) \equiv \{i \in N \mid \; \exists j \in N, ij \in g\}$ denote the set of connected players in the network g.

A path between players i and j in a network $g \in G$ is a sequence $i_1, \ldots, i_K$ such that $i_k i_{k+1} \in g$ for each $k \in \{1, \ldots, K-1\}$, with $i_1 = i$ and $i_K = j$.

A component of a network g, is a nonempty subnetwork g' of g such that :

(i) if $i \in N(g')$ and $j \in N(g')$ where $j \neq i$, then there exists a path in g' between i and j, and

(ii) if $i \in N(g')$ and $ij \in g$, then $ij \in g'$

Let $C(g)$ denote the set of all components of a network g. Notice that for all component g' in $C(g)$, and all pair of players $(i, j) \in N(g')$, the link between i and j may be direct or indirect. A component is then a set of completely connected players in g (some may be directly connected, others indirectly). For some purposes, people prefer to interact with directly linked partners. In a friendship network for example, a friend of my friend is not necessarily my friend. Thus, it is useful to study separately the direct links. Components fail to give an account of direct links. Therefore, we focus on the counterpart of components which only collects direct links, the *cliques*. These are the sets of completely *directly connected* players.

1.5.2 Link with covers

For each network g in G, define :

- $D(g)$ as the set of all directly connected elements of $N(g)$. Formally, $D(g) \equiv \{S \subseteq N(g) \mid \; \forall i, j \in S, ij \in g\}$.
- $\mathcal{C}l(g) \equiv \{S \in D(g) \mid \; \nexists S' \in D(g) \; s.t. \; S \subset S'\}$ as the set of completely directly connected elements of $N(g)$. $\mathcal{C}l(g)$ is the set of cliques of g.

– $I(g) \equiv N \setminus N(g)$ as the set of all singletons that have no link according to the network g.

– $\gamma_g \equiv I(g) \cup \mathcal{C}l(g)$.

Proposition 3. *For all $g \in G$, $\gamma_g \in \Gamma$ and is unique.*

Proof.

For the proof it is sufficient to notice that $\bigcup_{S \in \gamma_g} S = N$. Furthermore, for a fixed g, $\mathcal{C}l(g)$ is unique by definition, the same for $I(g)$. Thus γ_g is unique for any g □

Example 4. Let $N = \{1, 2, 3, 4, 5\}$ be the set of all players. Consider the following 5-person networks A, B, C. Table 1.5 verifies the proposition.

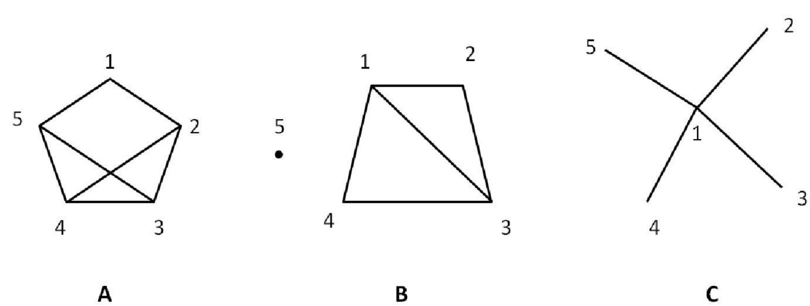

FIGURE 1.1 – 5-person networks

g	$\mathcal{C}l(g)$	$I(g)$	γ_g
A	$\{12, 15, 234, 345\}$	$\emptyset$	$\{12, 15, 234, 345\}$
B	$\{123, 134\}$	$\{5\}$	$\{123, 134, 5\}$
C	$\{12, 13, 14, 15\}$	$\emptyset$	$\{12, 13, 14, 15\}$

TABLE 1.5 – A cover of $g : \gamma_g$

Notice that $\gamma_A, \gamma_B, \gamma_C$ are all covers of N.

We name this decomposition from a network to a cover, *the cover representation.* The extreme cases of $\gamma_\emptyset$ and γ_N are respectively the coalition structures of singletons sets and the grand coalition.

Proposition 3 states that given a network g, the cover representation γ_g lies in Γ. But, is the converse also true ? To answer the question, we need additional definitions.

A *cycle* in a network $g \in G$ is path $i_1, \ldots, i_K$ where $i_1 = i_K$. A *circle* is a network that has a single cycle and is such that each node has exactly two neighbors. A network is said to be *composed of circles* if it is a circle or each of its components is a circle.

Let $\Gamma_0 \subset \Gamma$ denote the set of all covers γ_0 such that for all $S \in \gamma_0$, $|S| = 2$, and for all $i \in N$ there exists a unique couple $S, S' \in \gamma_0$ such that $S \cap S' = \{i\}$.

Proposition 4. *A network $g \in G$ is composed of circles if and only if $\gamma_g \in \Gamma_0$.*

Proof.

Let $\widetilde{g} \in G$ be a network composed of circles. Let g denote a typical component of $\widetilde{g}$. By definition, g is a circle. The circle g has a single path $i_1, i_2, ..., i_m, i_{m+1}$, with $i_{m+1} = i_1$. Let $S_j \equiv \{i_j, i_{j+1}\}, 1 \leq j \leq m$. The cover representation of g is $\gamma_g \equiv \{S_1, S_2, ..., S_m\}$. For all $j < m$, $|S_j| = 2$, $S_j \cap S_{j+1} = \{i_j\}$, $|S_m| = 2$, and $S_m \cap S_1 = \{i_1\}$. The same properties obtains for all components of $\widetilde{g}$. Since components are disjoint networks, then $\gamma_{\widetilde{g}} \equiv \bigcup_{g \in C(\widetilde{g})} \gamma_g$. Thus, $\gamma_{\widetilde{g}} \in \Gamma_0$.

Let $\gamma_0 \in \Gamma_0$ such that $\gamma_0 \neq \emptyset$. The set γ_0 is a cover of N of size n. By the uniqueness requirement in the definition of Γ_0, we obtain that γ_0 contains n distinct coalitions of size two each. The proof proceeds in rounds.

At the first round, choose $S_1 \equiv \{i_1, i_2\} \in \gamma_0$. Set S_2 as the coalition in γ_0 such that $S_2 \equiv \{i_2, i_3\}$ with $i_3 \neq i_1$. Continue step by step, choosing at step k, $S_k \equiv \{i_k, i_{k+1}\}$ such that $i_{k+1} \notin \{i_2, i_3, \ldots, i_{k-1}\}$. As N is finite, there exists a step m such that $i_{m+1} = i_1$. The path $i_1, i_2, ..., i_m, i_{m+1}$ is a cycle and the induced network is a circle. If $m = n$, then the proof is over. Else, $m < n$ and move to round two, with the set $N \setminus \{i_1, i_2, \ldots, i_m\}$ and proceed exactly as the first round.

Since N is finite, we construct step by step, the sets $S_1, S_2, \ldots, S_n$, with $S_n \equiv \{i_n, i_{n+1}\}$ and $i_{n+1} = i_j$, $1 \leq j \leq n$. Thus, the the path $i_1, i_2, ..., i_n, i_{n+1}$ defines either a cycle, or a succession of cycles. Thus, the network $\widetilde{g}$ obtained from this path is composed of circles and $\gamma_{\widetilde{g}} = \gamma_0$. □

Definition 6. *A cover $\gamma \in \Gamma$ is parsimonious if $\gamma = \{N\}$ or :*

(i) $S \nsubseteq S'$ for all distinct pairs of coalitions $S, S' \in \gamma$, and

(ii) there exists an ordering $i_1, i_2, \ldots, i_m$ of the indexes of coalitions in γ such that,
$S_{i_2} \setminus S_{i_1} \neq \emptyset$, $S_{i_3} \setminus \{S_{i_2} \cup S_{i_1}\} \neq \emptyset$,..., $S_{i_m} \setminus \{S_{i_{m-1}} \cup S_{i_{m-2}} \cup \ldots \cup S_{i_1}\} \neq \emptyset$, and
$\gamma = \{S_{i_1}, S_{i_2}, \ldots, S_{i_m}\}$.

Let Γ_{Pars} denote the set of all parsimonious covers of N.

Example 5. For $N = \{1, 2, 3, 4, 5\}$, let $\gamma = \{123, 235, 4\}$, $\gamma' = \{12, 123, 45\}$, and $\gamma'' = \{123, 234, 345, 451, 512\}$. According to the definition of Γ_{Pars}, $\gamma \in \Gamma_{Pars}$, $\gamma' \notin \Gamma_{Pars}$ because $12 \subset 123$, and $\gamma'' \notin \Gamma_{Pars}$ because each coalition is included in the union of the remaining.

Proposition 5. $\Gamma_{Pars} = \{\gamma_g : g \in G$ *and no component of* g *is a circle* $\}$.

Proof.

First, notice that if a component g' of g is a circle, $\gamma_{g'}$ is not parsimonious. Whatever ordering of the indexes of elements of $\gamma_{g'} \equiv \{S_1, S_2, \ldots, S_m\}$ that we consider, $S_1 \cup S_2 \cup \ldots \cup S_{m-1} = N(g')$. Therefore, $S_m \setminus \{S_{m-1} \cup S_{m-2} \cup \ldots \cup S_1\} = \emptyset$. Thus, γ_g is not parsimonious.

Second, consider a clique S_1 of g. Either $S_1 = N$ and γ_g is parsimonious, or there exists $i_1 \in S_1$ and $i_2 \in N \setminus S_1$ such that the link $i_1 i_2 \notin g$. Let S_2 be a clique of g that contains i_2. Either $S_2 \cup S_1 = N$, or there exists $i'_2 \in S_2$ and $i_3 \in N \setminus (S_2 \cup S_1)$ such that the link $i_3 i'_2 \notin g$. Let S_3 be a clique of g that contains i_3. Step by step, we construct an ordering of the indexes of the coalitions $S_1, S_2, \ldots$ that satisfy the parsimonious requirement (ii). As N is finite, there exists a step m such that $N \setminus \bigcup_{1\leq j\leq m} S_j = \emptyset$. Thus, $\bigcup_{1\leq j\leq m} S_j = N$.

If $\gamma_g = \{S_1, S_2, \ldots, S_m\}$, then γ_g verifies the requirement (ii) of parsimonious property. Else, suppose there exists $S \in \gamma_g$ such that $S \notin \{S_1, S_2, \ldots, S_m\}$. We test coalitions backward from S_m to S_1. We say that S_m pass the test if there exists $i \in S$ such that $i \in S_m$ and $i \notin \bigcup_{1\leq j\leq m-1} S_j$. Thus, $S_m \setminus S \neq \emptyset$, $S \setminus \bigcup_{1\leq j\leq m-1} S_j \neq \emptyset$, and $S_m \setminus (S \cup \bigcup_{1\leq j\leq m-1} S_j) \neq \emptyset$. In contrary, if S_m fails the test, then $S \subset \bigcup_{1\leq j\leq m-1} S_j$, and S_{m-1} takes the test. If S_{m-1} also fails the test, then we proceed with S_{m-2}. If coalitions in γ_g fail the test up to S_2, then S_2 can not fail the test, otherwise $S \subset S_1$ and this is not possible because S and S_1 are cliques. Therefore, there exists $k \in \{2, 3, \ldots, m\}$ such that $S_k \setminus S \neq \emptyset$, $S \setminus \bigcup_{1\leq j\leq k-1} S_j \neq \emptyset$, and $S_k \setminus (S \cup \bigcup_{1\leq j\leq k-1} S_j) \neq \emptyset$. Thus, rank the indexes of the coalitions as : for $1 \leq j < k-1$, $S_{i_j} \equiv S_j$; $S_{i_k} \equiv S$; and for $k \leq j \leq m$, $S_{i_{j+1}} \equiv S_j$. If $\gamma_g = \{S_{i_1}, S_{i_2}, \ldots, S_{i_{m+1}}\}$, then γ_g verifies the requirement (ii) of parsimonious property. Otherwise we proceed the same way as we have done previously with S until we have the ordering of all the coalitions of γ_g that satisfies the requirement (ii) of parsimonious property.

Furthermore, since cliques are sets of completely directly linked players, no inclusion is possible. Therefore, the parsimonious requirement (i) is satisfied. Thus, γ_g is parsimonious if no component of g is a circle.

Hence, $\{\gamma_g, g \in G, \text{ no component of } g \text{ is a circle } \} \subseteq \Gamma_{Pars}$.

Third, let $\gamma \in \Gamma_{Pars}$. Consider the ordering of the indexes of the coalitions of γ, $S_1, S_2, \ldots, S_m$ such that there exists $i_1 \in S_2 \setminus S_1$, there exists $i_2 \in S_3 \setminus \{S_2 \cup S_1\}$,..., there exists $i_{m-1} \in S_m \setminus \{S_{m-1} \cup S_{m-2} \cup \ldots \cup S_1\}$. None of these sets is empty. For all l such that $|S_l| \geq 2$, let $g/S_l = \{ij, i \neq j, i \in S_l, j \in S_l\}$. As γ is parsimonious, $i_l \in S_l$ and not in other $S_{l'}$, $l' < l$ because of the ordering of the indexes. Let $g = \bigcup_{l=1}^{m'} g/S_l$. The set g is a network and no component of g is a circle because

$S_2 \setminus S_1 \neq \emptyset$, $S_3 \setminus \{S_2 \cup S_1\} \neq \emptyset$,...,$S_m \setminus \{S_{m-1} \cup S_{m-2} \cup \ldots \cup S_1\}$. By parsimonious requirement (i), there is no inclusion among such sets S_l. By construction, each such S_l is a clique of g and $\gamma_g = \gamma$. □

Remark 5.

The set of networks composed of circles is represented by Γ_0. The set of networks that do no have a circle as component is represented by Γ_{Pars}. Between these two categories of networks, there are networks with some components being circles. For such networks, we can separate them into two subnetworks, g_1 and g_2. The subnetwork g_1 is the collection of all the components that are circles, and g_2 is the collection of remaining components. Thus, g_1 has a cover representation of the form of Γ_0, and g_2 has a cover representation of the form of Γ_{Pars}. Therefore, each network admits a cover representation. In contrast, covers have much more structure than networks. For example, covers such that there is inclusion between some coalitions do not represent any network. Furthermore, distinct covers may have the same network representation. For example, the complete network may represent N or the cover consisting of all coalitions of size 2.

The following theorem shows that some networks are "equilibrium networks" in the sense that their cover representations are equilibrium outcomes of the game that we have defined in the third section of the chapter.

Theorem 3. *If $g \in G$ is a network such that none of its components is a circle, then there exists a cover function (bargaining) game that admits γ_g as an equilibrium outcome.*

Proof.

For the proof, we need a triple (N, v, ρ) as defined in Section 3.

First, let N be a set of n players and g be a network on g such that g is not a circle. By Proposition 5, γ_g is parsimonious. Consider the ordering of $\gamma_g \equiv \{S_1, S_2, \ldots, S_m\}$, such that there exists $i_1 \in S_2 \setminus S_1$, there exists $i_2 \in S_3 \setminus \{S_2 \cup S_1\}$,...,there exists $i_{m-1} \in S_m \setminus \{S_{m-1} \cup S_{m-2} \cup \ldots \cup S_1\}$.

Second, define ρ by $\rho^p(N) = i_0$ where i_0 is the lowest indexed individual in S_1 ; $\rho^p(N \setminus S_1) = i_1$; $\rho^p(N \setminus \{S_2 \cup S_1\}) = i_2$; ... ; $\rho^p(N \setminus \{S_{m-1} \cup S_{m-2} \cup \ldots \cup S_1\}) = i_{m-1}$; and $\rho^p(S) = i_S$ where i_S is the lowest indexed individual in S for any other subset $S \subseteq N$. Let $\rho^r(S)$ be the increasing ordering of indexed individuals in $S \setminus \{\rho^p(S)\}$ for all $S \subseteq N$.

Third, define the value function by $v(S, \gamma) = 0$ for each embedded coalition (S, γ) such that $S \notin \gamma_g$ or $\gamma \neq \gamma_g$; $v(S_j, \gamma_g) = (m - j + 1)n$, for $j = 1, \ldots, m$.

Now we show that γ_g is a stationary subgame perfect equilibrium with no delay for the game defined by (N, v, ρ). An equilibrium action sequence is :

At stage $k = 1$ of the game, $\rho^p(N) = i_0$ makes the first proposal. She can no do better than proposing S_1. Suppose that i_0 proposes $S \notin \gamma_g$, this proposal will be rejected and the rejector will make another proposal in γ_g. Suppose that i_0 proposes another set in γ_g containing i_0 but not S_1, by proposing S_1, the expected payoff is the highest. Following i_0, all the responders in S_1 accept the proposal. In case of rejection, S_1 will not form or will form with a delay, and this strategy is strictly dominated by accepting. Thus, S_1 forms and the updated set of proposers is $N \setminus S_1$.

At stage $k = 2$ of the game, $\rho^p(N \setminus S_1) = i_1$. Notice that $i_1 \in S_2 \setminus S_1$. For the same reasons as at stage $k = 1$, i_1 can not do better than making an acceptable proposal S_2.

The game continue the same way up to stage $k = m$ and S_m forms.

Therefore no deviation from the previous strategy is profitable for the deviator. Thus the equilibrium outcome of the game is γ_g. □

By Theorem 3, each network such that none of its component is a circle can be obtained as equilibrium outcome of a bargaining process. This result is essential to explain network formation as a strategic process.

1.6 Conclusion

In this chapter, we study the formation of coalitions in a general framework where each player can belong to more than one coalition. We provide a model of bargaining game, namely cover function bargaining game, which allows the formation of overlapping coalitions. After showing the existence of an equilibrium with mild degree of mixing, we provide an algorithm to compute a no delay equilibrium in the symmetric case. Finally, we show that there exists a one-to-one link between network structures and overlapping coalitions structures. As a consequence, we show that each network such that none of its components is a circle is an equilibrium outcome of a bargaining process.

Chapitre 2

Games with overlapping coalitions and their cores

2.1 Introduction

Forming coalitions is a common behavior in politics, environmental issues, provision of public goods, customs unions, and many other economic, social, and political situations. Some coalitions are disjoint and others are overlapping. However, most of the economics literature on coalition formation, both applied and theoretical, is restricted to models where *coalitions cannot overlap*. Yet coalitions formed in Free Trade Agreements (FTAs), risk sharing, and environmental agreements are overlapping. For example, Norway is signatory to the European Free Trade Association but also has an FTA with the European Union and the Baltic states. In developing countries, informal risk sharing groups known as Rotating Saving and Credit Associations (ROSCAs) are overlapping. Also on environmental issues, the Asia-Pacific Partnership (APP) on clean development and climate is signed by Australia, Canada, China, India, Japan, South Korea and the United States of America. At the same time, the United States of America and Canada are signatories of the Convention on Long-Range Transboundary Air Pollution (LRTAP). These examples and many others show the importance of agreements involving overlapping coalitions in economic and social interactions. The aim of this chapter is to develop a cooperative game that accommodates overlapping coalitions.

Myerson, in 1980, introduced the idea of overlapping coalitions in the economics literature. However, the expression *overlapping coalition* has been well known in other disciplines like computer science and robotics. [1] The problems addressed there, however, are different from those of economic interest. In game theory, there are very few models where coalitions are allowed to overlap and there remains more to be done on that issue (D. Ray, 2007). Modeling coalition formation follows two approaches : the blocking approach and the bargaining approach. If attention is focused on non-cooperation, then individuals are treated as the fundamental behavioral units and the bargaining approach is taken [Rubinstein (1982), Chatterjee et al. (1993), Okada (1996), Ray and Vohra (1999) and so on]. To the best of our knowledge, one of the first papers to accommodate overlapping coalitions following the bargaining approach is Agbaglah (2011). On the other hand if attention is focused on cooperation, then coalitions are treated as the fundamental behavioral units and the blocking approach is taken. Following the blocking approach, solution concepts like the core, the Shapley value, the equilibrium binding agreement and others are developed in the literature. However, a few of these concepts accommodate overlapping coalitions : the difficulty resides essentially in the treatment of individuals who belong to more than one coalition. Albizuri et al. (2006) develop an extension of Owen value (and therefore Shapley value) to overlapping coalitions. In their paper, the number of distinct coalitions that an individual can belong to is limited. We do not have these limitations in this chapter. Chalkiadakis et al. (2008, 2010) introduce a notion of core for overlapping coalitions. In their setting, a coalition structure is a list rather than a set. Besides, there are coalition structures with infinite coalitions.

In this chapter we focus on the core (the most popular solution concepts) and we develop an extension to overlapping coalitions. The core was first defined in a

1. See Kraus et al. 1998, Hu et al. 2007, and Dang 2006 for more details.

characteristic function game, and thereafter extended (Thrall and Lucas, 1963) to partition function game. Several notions of core have been proposed in the literature. The $\alpha-$ and $\beta-$core (Aumann and Peleg, 1960), the $\omega-$core (Shenoy, 1979), the $\delta-$core (d'Aspremont et Al., 1983), the coalition structure core (Greenberg, 1994), the $\gamma-$core (Tulkens and Chander, 1997), the $r-$core (Huang and Sjostrom, 2003) are some of the most popular. The difference in these notions of core resides essentially in the way deviation and the reaction to deviation (residual players' behavior) are modeled.

To be more precise, we follow the spirit of the coalition structure core defined by Greenberg (1994), and specifically its extension to partition function games by Koczy (2007). This is the recursive core. It models reaction to deviation without imposing any behavioral assumption (optimism or pessimism). Following a deviation, the residual players choose any of their most preferred partitions. In that setting, *how does the recursive core behave in a setting where coalitions are allowed to overlap ? What concept of dominance is relevant in that setting ? What is the behavior of residual players ; especially those who belong to more than one coalition ?*

We answer these questions by organizing the chapter as follows. In Section 2.2 we present basic definitions. In Section 2.3 we allow coalitions to overlap, we define *cover function games*, and we develop an extension of the recursive core. In Section 2.4 we introduce a new notion of deviation and investigate its implications. Finally we conclude in Section 2.5.

2.2 Some definitions

Let $N \equiv \{1, 2, \ldots, n\}$ be the set of all players. A *coalition* S is a non-empty subset of N. A cover γ of N is a collection of coalitions, $S_1, S_2, \ldots, S_m$, such that $\bigcup_{k=1}^{m} S_k = N$: thus $\gamma \equiv \{S_1, S_2, \ldots, S_m\}$. Let γ_S denote a cover of a coalition S.

We denote by Γ the set of all covers of N and by Γ_S be the set of all covers of a coalition S. For a collection of coalitions λ, we denote by $\Gamma(\lambda)$ the subset of Γ such that $\forall\gamma \in \Gamma(\lambda),\ \lambda \subseteq \gamma$.

An *embedded coalition*[2] is a pair (S,γ) such that $S \in \gamma$ and $\gamma \in \Gamma$. The set of all embedded coalitions is denoted by Σ and defined by $\Sigma \equiv \{(S,\gamma) \mid S \in \gamma, \gamma \in \Gamma\}$.

In the remainder, we use the expression *overlapping coalition structure* to designate a cover whenever we want to emphasize on its structure. For simplicity of notation, we write a coalition explicitly $S = ijk...$ instead of $S = \{i, j, k, ...\}$.

Definition 7. *A cover function $v : \Sigma \to \Re$ is such that for all $(S,\gamma) \in \Sigma$, $v(S,\gamma) \geq 0$. A* cover function game *is the pair (N, v), where N is the set of players and v a corresponding cover function.*

In this definition, the cover function is nonnegative. This normalization is for simplicity and permits to have a minimum value of 0. After all what is needed is only a value function that is bounded below. Notice that v is defined only on embedded sets. Therefore, if (S,γ) is such that $S \notin \gamma$, then $v(S,\gamma) \equiv 0$.

Consider a cover function game (N, v), and a cover $\gamma \in \Gamma$.

An *attribution* $\widetilde{x} \equiv (x_i^S)_{i\in S,\ S\in\gamma}$ is a collection of nonnegative real numbers. Each such x_i^S represents the payoff that player i receives from coalition S. Given an attribution $\widetilde{x}$, the (overall) payoff awarded to player i is $x_i \equiv \sum_{S\ni i} x_i^S$.

An *outcome* is an ordered pair (x,γ) where $x \in \Re^N$, $x \equiv (x_i)_{i\in N}$, such that x_i is the payoff to player i . Notice that two attributions $\widetilde{x}$ and $\widetilde{x'}$ such that for all i, $x_i = x'_i$ yield the same payoff vector. Therefore, we focus in the remainder more on outcomes than attributions.[3] For all coalition S, let $x_S \equiv (x_i)_{i\in S}$.

An outcome (x,γ) is *feasible* if for all $S \in \gamma$, $\sum_{i\in S} x_i^S = v(S,\gamma)$. We denote by

2. We borrow this definition from Macho-Stadler et al. (2007).

3. Attributions have detailed information about the composition of payoffs. However, we decide (for simplicity) to mention payoffs only because, after all, players base their decisions on payoffs which are aggregated information.

$\Omega(N, v)$ the set of all the feasible outcomes in the game (N, v).

In this chapter, we aim to develop a cooperative game that accommodates overlapping coalitions. For this purpose, outcomes of the game should be such that some players may belong to more than one coalition. Formally, given a cover function game (N, v), a player $i \in N$ is an *overlapping player* if the cover function v is such that $v(S, \gamma) > 0$ and $v(S', \gamma) > 0$ for at least two distinct coalitions S and S' containing i and a least one cover $\gamma \in \Gamma$. [4] We denote by O_S the set of all overlapping players in a coalition S. For $x, x' \in \Re^N$ ($x = (x_i)_{i \in N}$, $x' = (x'_i)_{i \in N}$) and $S \subseteq N$, $x >_S x'$ if $x_i \geq x'_i$ for all $i \in S$, and $x_i > x'_i$ for some $i \in S$.

2.2.1 Deviation

For a better definition of a core in a setting where coalitions may overlap, we need first to clearly define what "deviation" means. The literature on coalition formation admits that given a coalition structure, players deviate with the intention to form new coalitions that yield them better payoffs (for at least one player or for all depending on the approach). The consequence of any deviation is the formation of a new coalition structure. On the other hand, there cannot be a new coalition structure without deviations. Thus, deviation and formation of a new coalition structure are equivalent notions. With that spirit, given an overlapping coalition structure, we define deviation as equivalent to the formation of a new overlapping coalition structure. We allow individual as well as set deviations. A set $D \subseteq N$ of players deviates with the intention to form a cover γ_D of D [5]. This deviation *induces the formation of a new overlapping coalition structure*. In our setting we introduce two notions of deviation to accommodate overlapping coalitions : *total deviation* and *partial deviation*. A total deviation occurs when a deviating coalition "disappears" after the deviation ; this is possible if the deviating coalition breaks up or merges with another one. In contrast, a partial

4. The game is non-negative by normalization and therefore the minimum possible value is 0.
5. Once we define formally a set deviation, individual deviation is obtained when the deviating set is a singleton.

deviation happens without dislocation of deviating coalitions ; this is possible only for coalitions containing overlapping players. The following example will help to fix ideas.

Example 6.

Consider the following 5-person cover function game.

$N = \{1, 2, 3, 4, 5\}, \gamma = \{12, 34, 52\}, v(12, \gamma) = 1, v(52, \gamma) = 10$, and

$v(S, \gamma) = 0$ for any other $(S, \gamma) \in \Sigma$.

Player 2 is an overlapping player in the game (N, v) because, $v(12, \gamma) > 0$ and $v(52, \gamma) > 0$.

Consider the following covers : $\gamma' = \{12, 345\}$; $\gamma'' = \{12, 34, 5\}$; $\gamma''' = \{12, 34, 52\}$.

In a deviation from γ' to γ'', player 5 performs a total deviation because she leaves the coalition 345 and this coalition is no more in γ''.

In contrast, if we consider a deviation from γ'' to γ''', player 2 performs a partial deviation. The coalition 12 that contains player 2 is maintained in the new cover γ'''. This partial deviation is performed by player 2 who is an overlapping player.

Formally, let $\gamma, \gamma' \in \Gamma$ and consider a deviation from γ to γ'.

Definition 8.

1. ***Total deviation : all players.***

   A player $i \in N$ performs a total deviation if and only if :

   there exists an embedded coalition (S, γ) such that $i \in S$ and $S \notin \gamma'$. [6]

2. ***Partial deviation : only overlapping players.***

   A player $i \in N$ performs a partial deviation if and only if :

   i is an overlapping player and for all embedded coalition (S, γ), if $i \in S$, then $S \in \gamma'$.

6. All coalitions that contain some total deviating players break up in the new cover γ'. This is the traditional notion of deviation.

3. ***Set deviation****.*

A player set $D \subseteq N$ *deviates to form a cover* γ_D *of* D *if and only if :*

each player $i \in D$ *performs a deviation (either partial or total).*

Remark 6.

- The notion of set deviation is a collection of multiple deviations, some are total and others are partial.
- In our setting, a deviation may affect the payoffs of all players, including non-deviators. Due to these externalities, if an overlapping player partially deviates from a single coalition to more than one coalition, the payoff of the players in the former coalition may be affected. But, she decides to deviate only if her overall payoff from the new cover (obtained after deviation) is higher than previously.
- In the definition, we do not specify what happens to the remaining players if some deviate. The following section focuses on that aspect.

In the sequel, we model the behavior of *residual players*. These are non-deviators who are affected by a deviation (due to externalities) and may react to it.

2.2.2 Residual game

Since a deviation may affect the payoffs of all players, if a player set $D \subseteq N$ deviates, the remaining players in $N \setminus D$ may adjust to it by reacting. For the reasons of consistency, they will also play a game, similar to the stage game, called the *residual game*. In the existing literature, the residual game concerns players in $N \setminus D$ following a deviation by D. In this section we follow this traditional approach as a benchmark. Later in this chapter, we propose another concept of residual game that is more adapted to overlapping coalition structures.

Suppose that a player set $D \subseteq N$ performs a deviation. Players in D form a cover γ_D of D. Let $R \equiv N \setminus D$ be the set of residual players. Consider embedded coalitions of the form (S, γ_R) where $S \subseteq R$ and $\gamma_R \in \Gamma_R$ a cover of R. Let Σ_R be the collection of all such embedded coalitions.

Definition 9. *The residual game is the game* (N, v_{γ_D}) *where* v_{γ_D} *is such that for all* $(S, \gamma_R) \in \Sigma_R$*,* $v_{\gamma_D}(S, \gamma_R) \equiv v(S, \gamma_R \cup \gamma_D)$ *.*

Remark 7.

1. In this definition, $\gamma_R \cup \gamma_D$ is a cover of N.
2. If $Q \notin \gamma_R, v_{\gamma_D}(Q, \gamma_R) = 0$.
3. The residual game is a cover function game on its own.
4. This definition implicitly says that once players deviate and form a new cover, they commit to this cover. There will not be further deviation performed by these players.
5. If there are no overlapping players (if we consider only partitions instead of covers), this definition is equivalent to the one in the partition function case, proposed by Koczy (2007).

2.2.3 Dominance

In the following, we define dominance in two notions, the optimistic dominance and the pessimistic dominance.

Definition 10.

Optimistic dominance :

An outcome (x, γ) *is dominated via a coalition* S *inducing the formation of a cover* γ_S *of* S*, if for* ***at least one*** *outcome* (x'_R, γ_R) *from the residual game, with* $R \equiv N \setminus S$*, there exists a* ***feasible outcome*** $((x'_S, x'_R), \gamma')$ *, with* $\gamma' \equiv \gamma_S \cup \gamma_R$*, such that* $x' >_S x$*.*

Pessimistic dominance :

An outcome (x, γ) *is dominated via a coalition* S *inducing the formation of a cover* γ_S *of* S*, if for* ***all*** *outcomes* (x'_R, γ_R) *from the residual game, with* $R \equiv N \setminus S$*, there exists a* ***feasible outcome*** $((x'_S, x'_R), \gamma')$*, with* $\gamma' \equiv \gamma_S \cup \gamma_R$*, such that* $x' >_S x$*.*

Remark 8.

According to the optimistic dominance, a coalition deviates as soon as there exists an outcome from the residual game that make the deviators better : even if it is not sure residual players will allow that. Whereas from the pessimistic dominance perspective, a coalition deviates only if whatever strategy the residuals play, the deviators will be better off. One can see that optimistic dominance will be very easy to obtain, but pessimistic dominance will be very hard to obtain.

2.3 Extended recursive core

To define the extended recursive core we set the grand coalition $\{N\}$ as the starting point. Discussion starts from the grand coalition. From there, a player set S may be better off by deviating and forming a cover γ_S of S. Following this deviation, the remaining players may react strategically by playing the residual game. Remember that once γ_S forms, players in S commit to it. Thus, no player in S may accomplish further deviations. This assumption is also made by Koczy (2007). We need this assumption to build the recursion as the residual game is a game on its own, defined on the complement of S, with a **fewer** number of players than the initial game. Thus, we define the core recursively on the number of players that can perform further deviations.

2.3.1 Optimistic, pessimistic core

Consider the game (N, v), and let $n \equiv |N|$.

- For a game with only one player, say $\{1\}$, the core is $C(\{1\}, v) \equiv \big\{v(\{1\}, \{1\}); \{1\}\big\}$.
- Assume that the core $C(R_k, v_k)$ is defined for every k-person game, $k = 1, 2, \ldots, n-1$. If the core is empty, define $A(R_k, v_k) \equiv \Omega(R_k, v_k)$,[7] otherwise, define $A(R, v) \equiv C(R_k, v_k)$.
- Now consider the game (N, v). An outcome (x, γ) is dominated via a coalition S inducing the formation of a cover γ_S of S, if for **at least one** outcome (resp.

7. Remember that $\Omega(R_k, v_k)$ is the set of all feasible outcomes of the game (R_k, v_k).

all outcomes) $(x'_R, \gamma_R) \in A(R, v_{\gamma_S})$, with $R \equiv N \setminus S$, there exists a **feasible outcome** $\big((x'_S, x'_R), \gamma\big)$, with $\gamma \in \Gamma(\gamma_S)$, such that $x' >_S x$.

– The optimistic (resp. pessimistic) core of an n-person game, $C_+(N, v)$ $\big($resp. $C_-(N, v)\big)$, is the set of non-dominated outcomes by optimistic dominance (resp. by pessimistic dominance).

Example 7. We denote by (N, v) the following 5-person cover function game, with $N = \{1, 2, 3, 4, 5\}$.

For simplicity of the notation we write $v(\gamma)$, instead of $v(S, \gamma)$.

$v(1, 2, 3, 4, 5) = (0, 0, 1, 0, 0)$

$v(12345) = (5)$

$v(12, 3, 45) = (2, 2, 2)$

$v(14, 3, 25) = v(15, 3, 24) = (1, 2, 1)$

$v(123, 45) = (9, 3)$

$v(12, 345) = (3, 9)$

$v(123, 345) = (8, 8)$

$v(1245, 3) = (2, 2)$

$v(\gamma) = 0$, for any other cover γ.

From our computations [8] the optimistic and pessimistic core coincide in this example. We have :

$C_+(N, v) = C_-(N, v) = \{((x_1, x_2, x_3, x_4, x_5); (123, 345))$ such that

8. Details are available upon request to the author.

$$\begin{cases} x_4 + x_5 \geq 3 \\ x_1 + x_2 \geq 3 \\ x_3 \geq 2 \\ x_1 + x_2 + x_3 \geq 9 \\ x_3 + x_4 + x_5 \geq 9 \\ x_1 + x_2 + x_3^{123} = 8 \\ x_3^{345} + x_4 + x_5 = 8 \\ x_3^{123} + x_3^{345} = x_3 \end{cases}$$

Even if it seems very normative, the cover function game in the example describes the situation of an informal insurance group. [9] Players have random income in village economies (in developing countries). If players are on their own, they do not get any insurance. If they come all together, they get few insurance due to heterogeneities (conflicts for example). If they form homogenous groups, [10] they get better insurance : for instance (12) and (45). Suppose that player 3 is an "elder" in the village with high social privileges and every informal insurance group is better off having 3. Notice that in this example, player 3 as "elder" has the ability to be an overlapping player.
The game exhibits externalities. For example if coalition (12) forms, her value depends on the remaining players. Consider a deviation from the cover $\{12, 345\}$ to $\{12, 3, 45\}$ (we have $v(12, 345) = (3, 9)$ and $v(12, 3, 45) = (2, 2, 2)$). Even if coalition (12) does not participate to the deviation, her total payoff of 3 decreases to 2. Therefore, players 1 and 2 may react to this deviation.

Remark 9.

1. Only player 3 in an "overlapping player" in this game.
2. The core is not empty in this example. For example $((3, 3, 4, 3, 3); \{123, 345\})$ is an outcome in the core.

9. It is empirically proved that risk sharing groups in agrarian villages are overlapping coalitions. For example, De Weerdt and Dercon (2005) observe on data from Tanzanian villages that insurance networks of villagers mostly overlap.
10. The homogenous groups may be religion, friendship, kinship, ethnicity, social class and so on.

3. An equilibrium overlapping coalition structure is $(123, 345)$.

2.3.2 Properties

In this section we derive properties of this new notion of core that accommodates overlapping coalitions as we see from the previous example.

Proposition 6. *The optimistic and pessimistic cores coincide with these notions on partition function games.* [11]

Proof.

Consider a cover function game (N, v) such that for all cover $\gamma \in \Gamma$, and all pairs of distinct embedded coalitions (S, γ) and (S', γ),
$S \cap S' \neq \emptyset \Rightarrow v(S, \gamma) = v(S', \gamma) = 0$.
For all cover $\gamma = \{S_1, S_2, ..., S_m\}$, let $\mathcal{S}_\gamma \equiv \{S_j \in \gamma \text{ such that } S_j \cap S_k \neq \emptyset \text{ for } j \neq k\}$. Let $\gamma \backslash \mathcal{S}_\gamma = \{S'_1, S'_2, ..., S'_l\}, l \leq m$. Let $S'_{l+1} \equiv \bigcup_{S_j \in \mathcal{S}_\gamma} S_j$ and $\Pi_\gamma = \{S'_1, S'_2, ..., S'_l, S'_{l+1}\})$. Thus, Π_γ is a partition of N. Define for all $S \in \Pi_\gamma$, $v'_\gamma(S, \Pi_\gamma) = v(S, \gamma)$. Let (N, v') denote the obtained partition function game. If an outcome (x, γ) is non dominated in the cover function game (N, v), the corresponding outcome (x', Π_γ) is non dominated in the partition function game (N, v'). Thus, our core is exactly the recursive core as in partition function game. By Lemma 10 (Koczy, 2007), [12] we conclude that our notion is a generalization of the coalition structure core. $\square$

Theorem 4. *For a cover function game (N, v), $C_+(N, v) \subseteq C_-(N, v)$*

Proof.

This is obvious because $\{mboxoptinisticdeviations\} \supseteq \{$ pessimistic deviations $\}$ $\square$

11. The coalition structure core is defined for characteristic function games whereas the core here is defined for cover function games.

12. Koczy (2007) Lemma 10 : Consider a partition function game (N, V) and a characteristic function game (N, v). If for all partition P and all coalition $S \in P$, $V(S, P) = v(S)$, then the recursive core and the coalition structure core coincide

From Theorem 4, we obtain a set interval of outcomes where the lower bound is $C_+(N,v)$ and upper bound $C_-(N,v)$. We define the *extended recursive core* as the set of outcomes lying between $C_+(N,v)$ and $C_-(N,v)$.

2.4 The overlapping coalition structure core

In this section we develop a notion of residual game that is more consistent with the stage game than the previous notion. The *consistent residual game* consists of a game played by residual players not restricted to the complement of the set of deviators. Instead, the set of residual players in this section intersects the set of deviators via overlapping players. Following a deviation by a player set D, we allow overlapping players in D to still play a role in the residual game. Even if they commit to their previous coalitions, they may write agreements with players in the complement of D. For this purpose, we introduce two kinds of players : *active* and *passive* players.

Definition 11. *Suppose that a player set D, performs a deviation and forms a cover γ_D of D.*

The set $N \setminus D$ consists of active players in the consistent residual game.

The set of overlapping players in D, O_D, consists of passive players in the consistent residual game.[13]

Suppose that a player set $D \subseteq N$ performs a deviation that yields the formation of a cover γ_D of D. Let $R \equiv (N \backslash D) \cup O_D$, and $\Lambda_R \equiv \{\lambda_R,$ collection of coalitions of R, such that $\lambda_R \cup \gamma_D \in \Gamma\}$. Consider embedded coalitions (S, λ_R) where $S \subseteq R$ and $\lambda_R \in \Lambda_R$. Let Σ_R be the collection of such embedded coalitions.

13. We use the expression passive because of the fact that even if they can write an agreement with active players, passive players do not initiate this agreement since they cannot write agreements on their own within O_D. They need at least one active player to do so.

Definition 12. *The consistent residual game is the game* (N, v_{γ_D}) *where* v_{γ_D} *is such that for all* $(S, \lambda_R) \in \Sigma_R$, $v(S, \lambda_R) \equiv v(S, \lambda_R \cup \gamma_D)$.

Remark 10.

1. In this definition, $\lambda_R \cup \gamma_D$ is a cover of N.
2. If $Q \notin \lambda, v_{\gamma_D}(Q, \lambda) = 0$.
3. This definition of residual game is consistent with the fact that even if overlapping players perform passive deviations, the cover γ_D that forms during the deviation of D remains unchanged in the following cover of N that will form. This definition is consistent because it allows overlapping individuals to continue using strategically their ability. Thus, we define the residual game on a larger set than the complement of D because some of the players that react to a deviation, can strategically conclude further agreements with some deviating overlapping players. This assumption holds because the cover function game is a TU game, and side payments are possible.

Proposition 7. *The consistent residual game is a cover function game on its own, defined on the set of active and passive players.*

Proof.

Suppose that a player set D deviates and form γ_D. Define the prolongation of the consistent residual game on $\mathbf{2}^R \times \Gamma_R$ where Γ_R is a cover of R. Consider $\lambda_R \in \Lambda_R$. Let $\lambda_R \equiv \{S_1, S_2, ..., S_k\}$.

If $\bigcup_{i=1}^{k} S_1 = N \setminus D$, then there is no passive player. Let $S_{k+1} \equiv O_D$. Thus, $\gamma_R \equiv \{S_1, S_2, ..., S_k, S_{k+1}\}$ is a cover of R. Let $\widetilde{v}_{\gamma_D}(S_{k+1}, \gamma_R) \equiv 0$.

If $N \setminus D \subset \bigcup_{i=1}^{k} S_1$, then there are some passive players. In this case let $S_{k+1} \equiv R \setminus \bigcup_{i=1}^{k} S_1$. Thus, $\gamma_R = \{S_1, S_2, ..., S_k, S_{k+1}\}$ is a cover of R. Let $\widetilde{v}_{\gamma_D}(S_{k+1}, \gamma_R) \equiv 0$.

For any other embedded coalition $(S, \lambda_R) \in \Sigma_R$, $\widetilde{v}_{\gamma_D}(S, \gamma_R) \equiv v_{\gamma_D}(S, \lambda_R)$.

The game $(R, \widetilde{v}_{\gamma_D})$ is a cover function game. □

2.4.1 Dominance

With the new definition of a residual game, we need to redefine the concept of dominance. The difference resides essentially in the sets of interest.

Definition 13.

Optimistic dominance

An outcome (x, γ) is dominated via a coalition S inducing the formation of a cover γ_S of S, if for **at least one** *outcome $((x'_{N\backslash S}, x'_{O_S}), \lambda_R)$ from the consistent residual game, there exists a* ***feasible outcome*** *$((x'_S, x'_{N\backslash S}), \gamma')$, $\gamma' \equiv \gamma_S \cup \gamma_R$, such that $x' >_S x$.*

Pessimistic dominance

An outcome (x, γ) is dominated via a coalition S inducing the formation of a cover γ_S of S, if for **all** *outcomes $((x'_{N\backslash S}, x'_{O_S})\lambda_R)$ from the consistent residual game, there exists a* ***feasible outcome*** *$((x'_S, x'_{N\backslash S}), \gamma')$, with $\gamma' \equiv \gamma_S \cup \gamma_R$, such that $x' >_S x$.*

Remark 11. The idea behind optimistic and pessimistic dominance here is the same as previously. What is different here is that an outcome from a residual game for overlapping players in S is not sufficient to determine their overall payoff. One need to add their payoff from the coalitions in γ_S.

As in section 2.3, once a cover γ_S of S forms, players may commit to it. Therefore, all further total deviation is prohibited (since total deviations break up already formed coalition in γ_S). But passive players in S may strategically perform further partial deviations. The core will be defined recursively on the number of active players as passive players can not deviate without active players.

2.4.2 Optimistic, pessimistic core

Consider the game (N, v), and let $n \equiv |N|$.

– For a game with only one player, say $\{1\}$, the core is $C^*(\{1\}, v) = \big\{v(\{1\}, \{1\}); \{1\}\big\}$.

- Assume that the core $C^*(R_k, v_k)$ is define for every game with k **active players**, $k = 1, 2, \ldots, n-1$. If the core is empty, define $A^*(R_k, v_k) \equiv \Omega(R_k, v_k)$, otherwise, define $A^*(R, v) \equiv C^*(R_k, v_k)$.
- An outcome (x, γ) is dominated via a coalition S inducing the formation of a cover γ_S of S, if for **at least one** outcome (resp. **all** outcomes) $\big((x'_{N\setminus S}, x'_{O_S}), \lambda_R\big)$ from $A^*(R, v_{\gamma_S})$, with $R \equiv (N \setminus S) \cup O_D$, there exists a **feasible outcome** $\big((x'_S, x'_{N\setminus S}), \gamma'\big)$, with $\gamma' \in \Gamma(\gamma_S)$ such that $x' >_S x$.
- The optimistic (resp. pessimistic) core of an n-person game, $C^*_+(N, v)$ $\big($resp. $C^*_-(N, v)\big)$, is the set of non-dominated outcomes by optimistic dominance (resp. by pessimistic dominance).

2.4.3 Properties

Previous results as the generalization result or the comparison between the optimistic and the pessimistic core hold. Furthermore, we compare the two concepts of core at the end of this section.

Remark 12. *The optimistic and pessimistic cores coincide with these notions on partition function games.*

Theorem 5. *For a cover function game (N, v), $C^*_+(N, v) \subseteq C^*_-(N, v)$.*

Proof.

The theorem holds for the same reasons as for Theorem 4. □

Once again Theorem 5 induces a set interval for outcomes, with the lower bound being $C^*_+(N, v)$ and the upper bound $C^*_-(N, v)$ in the inclusion sense. Therefore, we define a new concept of core that is more adapted to the setting of overlapping coalitions.

Definition 14. *The overlapping coalition structure core is the set of outcomes laying between $C^*_+(N, v)$ and $C^*_-(N, v)$.*

So far, we firstly define naively the residual game as a benchmark. Thereafter, we develop a more consistent notion of residual game. In the sequel we compare the two concepts of core that obtain under the two notions of residual games.

Theorem 6. *For a cover function game* (N,v),

$$C_+(N,v) \subseteq C^*_+(N,v) \subseteq C^*_-(N,v) \subseteq C_-(N,v)$$

Remark 13. With this consistent concept of core we obtain a refinement of the extended recursive core. As players become more sophisticated, the set of non-dominated outcomes shrinks.

Lemma 7. $C_+(N,v) \subseteq C^*_+(N,v)$

Proof Lemma 7.

Let us make the proof by contraposition. We will show that for all outcome (x,γ) such that $(x,\gamma) \notin C^*_+(N,v)$, then $(x,\gamma) \notin C_+(N,v)$. If $(x,\gamma) \notin C^*_+(N,v)$, then there exists $S \subseteq N$ and a payoff vector (y_{N-S}) for active players from $A^*_+(R, v_{\gamma_S}); R = (N-S) \cup O_S$, and a feasible outcome (y_S, y_{N-S}, γ') ; such that $y >_S x$. Notice that (y_{N-S}) comes from some $(y_R, \gamma_R) \in A^*_+(R, v_{\gamma_S})$. Two cases are possible :

Case 1 : No overlapping player in the set $\{i \in S | y >_S x\}$

Then (y_{N-S}) is a payoff vector from $A_+(N-S, v_{\gamma_S})$ (because $O_S = \emptyset$). Thus (y_S, y_{N-S}, γ') is a feasible outcome and $y >_S x$. So $(x,\gamma) \notin C_+(N,v)$.

Case 2 : At least one overlapping player in the set $\{i \in S | y >_S x\}$

Decompose $\lambda = \{S_1, S_2, ..., S_k\}$ in two sets : $\lambda_1 \cup \lambda_2$ such that : $\lambda_1 = \{S_i \in \lambda$ such that $S_i \cap O_S \neq \emptyset\}$; and $\lambda_2 = \{S_i \in \lambda$ such that $S_i \cap O_S = \emptyset\}$. Note O, the union of all the sets in λ_1, and D the set $N \setminus (S \cup O)$.

In the following we show that the set $S \cup O$ may deviate by forming the specific cover

$\gamma_S \cup \lambda_1$ of $S \cup O$.

Considering that (y_{N-S}) is a payoff vector from $A^*_+(R, v_{\gamma_S})$ (non dominated in a game with active players belonging to $N \setminus S$) and $D \subset N \setminus S$, we have $(y_D, \lambda_2) \in A_+\big(N - (S \cup O), v_{\gamma_{S\cup O}}\big)$. Furthermore, consider a payoff distribution where an overlapping player $i \in S$ gets x_i and the difference $d_i = y_i - x_i$ (which is nonnegative). Consider the sets $S_l \in \lambda_1$ that player i belongs to. For each player $j \in S_l \cap (O \setminus O_S)$ (active players belonging to S_l) give a fraction (for example equal fraction α ; but any distribution of nonnegative weights that sum to one will work) of d_i plus x_j. Do this for all overlapping players in O_S and all for all the active players in O. We have a distribution of payoffs y' with the following characteristics :

for all player $i \in S \setminus O_S$, $y'_i = y_i$

for all player $i \in O_S$, $y'_i = x_i$

for all player $i \in D$, $y'_i = y_i$

for the active players in O, $y'_i = x_i + \alpha d_i$

Thus, (y', γ') is feasible and $y' >_{S\cup O} x$. Hence $(x, \gamma) \notin C_+(N, v)$. □

Lemma 8. *$C^*_-(N, v) \subseteq C_-(N, v)$.*

Proof Lemma 8.

We proceed by contraposition. We will show that for all outcome (x, γ) such that $(x, \gamma) \notin C_-(N, v)$, then $(x, \gamma) \notin C^*_-(N, v)$.

If $(x, \gamma) \notin C_-(N, v)$, then there exists $S \subseteq N$ such that for all payoff vector (y_{N-S}) from $A_-(R, v_{\gamma_S})$, we have $y >_S x$. Consider one of these payoff vectors , (y^0_{N-S}), for active players from $A^*_-(R, v_{\gamma_S})$: notice that (y^0_{N-S}) comes from some $(y^0_R, \gamma_R) \in A^*_-(R, v_{\gamma_S})$ with $R \equiv (N - S) \cup O_S$. Two cases are possible :

Case 1 : No overlapping player in the set $\{i \in S | y^0 >_S x\}$

Then (y^0_{N-S}) is a payoff vector from $A_-(N - S, v_{\gamma_S})$. Thus $(y^0_S, y^0_{N-S}, \gamma')$ is a feasible outcome and $y^0 >_S x$. So $(x,\gamma) \notin C_-^*(N,v)$.

Case 2 : At least one overlapping player in the set $\{i \in S | y^0 >_S x\}$

Decompose $\lambda = \{S_1, S_2, ..., S_k\}$ in two sets : $\lambda_1 \cup \lambda_2$ such that : $\lambda_1 = \{S_i \in \lambda$ such that $S_i \cap O_S \neq \emptyset\}$; and $\lambda_2 = \{S_i \in \lambda$ such that $S_i \cap O_S = \emptyset$. Note O, the union of all the sets in λ_1, and D the union of all the sets in λ_2. Consider the deviation by $S \cup O$ forming a cover $\gamma_S \cup \lambda_1$. Because of the fact that $(y_R^0, \gamma_R) \in A_-^*(R, v_{\gamma_S})$, then $(y_D^0, \gamma_D) \in A_-\big(N - (S \cup O), v_{\gamma_{S\cup O}}\big)$. Furthermore consider the corresponding feasible outcome (y^0, γ') . We have : $(y^0, \gamma') = (y_{S\cup O}^0, y_D^0, \gamma')$ is feasible and $y^0 >_{S\cup O} x$. Hence, $(x,\gamma) \notin C_-^*(N,v)$. □

Proof Theorem 6.

From Lemmata 7 and 8, the proof of the theorem is straightforward. □

2.5 Conclusion

In this chapter, we develop an extension of the recursive core to cover function games. These are coalition formation games with externalities and overlapping coalition structures. We firstly follow the traditional view by defining residual players as separated from deviators. We show that the induced extended recursive core is a generalization of Greenberg's coalition structure core. Furthermore, we take an eclectic view by introducing a consistent concept of residual game where deviating overlapping players may set additional agreements (if they pleased) with residual players without reneging on their previous agreements. We show that the induced overlapping coalition structure core is not only a generalization of the coalition structure core, but also stands as a refinement of the extended recursive core.

Chapitre 3

Informal insurance : an approach by overlapping coalitions

3.1 Introduction

The need for insurance is tremendous in agrarian economies [1] because individual incomes are random. Outputs are precarious. They are subject to climate (farmers, breeders), to chance (fishers, hunters, workers in informal sector), to diseases, to wild animal invasions, and to destruction by fire. However, access to formal credit and insurance markets is almost non-existent. In that environment, individuals develop *informal insurance* [2] as a coping strategy. This is a reciprocal transfer agreement built to avoid negative consequences of idiosyncratic income shocks on personal consumption. Insurance is said to be informal because there is no signed paper, no collateral, and no legal court.

The economics literature, either theoretical or empirical, models informal insurance from two perspectives : the *group approach* and the *network approach*. The group approach models informal insurance as an arrangement at village level where individuals are organized as "clubs" [Kimball (1988), Coate and Ravallion (1993), Ligon et al

1. See Postner (1980) for characteristics of these economies.
2. See for instance Rosenzweig (1989), Udry (1990,1994), Townsend (1994)

(2002), Genicot and Ray (2003), and so on]. However, empirical findings are puzzling. Contrary to the predictions of this approach, there is no evidence for full risk sharing at village level. [3] Instead, informal insurance is organized between homogenous groups of individuals in the same village. These are kinship, neighborhood, friends, relatives, clans, castes, and families. These findings suggest to study informal insurance, using an approach based on networks. [4] This is known as the network approach. Transfers occur via social networks, modeled as a collection of pairs of individuals [De Weerdt and Dercon (2006), Fafchamps and Gubert (2007), Bramoullé and Kranton (2007), Bloch, Genicot and Ray (2008), Ambrus et al. (2010) and so on].

Based on empirical findings, we state in this chapter that these two approaches should be neither studied separately, nor viewed as opposite. Even if agreements may be negotiated one to one in homogenous networks, people neither live nor act alone. Decisions involve groups. In this respect, informal insurance is much more multilateral arrangements within homogenous and overlapping (de Weerdt and Dercon, 2006) groups. Thus, this chapter introduces two levels of rigidity to large insurance group sizes. At first level, we state that there exists a network prior to any insurance negotiation. Insurance can only take place between linked partners. Most of the time, this exogenous preexisting network reflects trust relationships. The second level consists of strategic reasons. For example, in some villages in developing countries, farmers are perpetually in conflict with breeders. In this context, risk sharing groups are organized within the community of farmers and also within the community of breeders. However, some few households belong to risk sharing groups in both communities : these are *overlapping individuals*. The multilateral approach to model informal insurance is at the best of our knowledge, the first attempt to merge the two approaches and to introduce the possibility for risk sharing groups to overlap.

3. See for example, Deaton (1992) with data from Ghana and Thaïland, Grimard (1997) with data from Côte d'Ivoire, Lund and Fafchamps (2003) with data from Philippines. The only exception at our knowledge is the paper by Townsend (1994) who shows the existence of almost full insurance with data from Indian villages

4. Some authors argue that the limited size of insurance groups is due to the cost of links ; Murgai et al. (2002)

As preexisting networks are basis of insurance negotiations[5], a question comes to mind : *under what conditions this preexisting network perpetually conveys transfers without any defection ?* If each individual commits to the agreement, the network will be viable. Thus, our research question posits the problem of commitment and this chapter can be viewed as a contribution to the literature on self-enforcing insurance arrangement, pointed out since 1980 by Postner and established at first in Kimball (1988). Commitment is enforced by a punishment following any defection. Thus, a viable agreement needs not only to be stable, but also self-enforcing : *if an individual defaults, the punishment following her action should be so severe that she is discouraged to do so.* The literature is not unanimous on how to model punishment. According to the group approach, punishment should be total exclusion of the deviator. *This is a very strong punishment.* In contrary the network approach proposes the severance of only the link between the deviator and the victim. *This is a very weak punishment.* Bloch, Genicot and Ray (2008) departs from these two extreme punishment schemes by proposing the "level-q" punishment. This punishment depends on the number of links but the extent "q" of the punishment is exogenous.

Without exposing here the model, we propose to fix ideas by introducing the following illustration (Figure 3.1) of a 7-person village. Suppose that trust relationships within the village depicts the following network g.

The basic structure of risk sharing groups is defined from g and consists of the sets of completely directly connected individuals[6]. Coalitions that represent risk sharing groups are $S_1 = \{1\}$, $S_2 = \{2,3,4\}$, $S_3 = \{4,5,6\}$, $S_4 = \{4,5,7\}$. Thus, informal insurance groups consist of the overlapping coalition structure $\gamma \equiv \{S_1, S_2, S_3, S_4\}$. Suppose that for some reason, 5 contemplates to deviate to 6.

5. In most of the papers, negotiation is supposed to commence at the village level, meaning that every individual can build an agreement with any other one if there is an economic advantage to do so. We take an opposite view here by stating that only individuals who trust each other can negotiate risk sharing agreements.

6. See Agbaglah (2011) for details about this overlapping coalition structure.

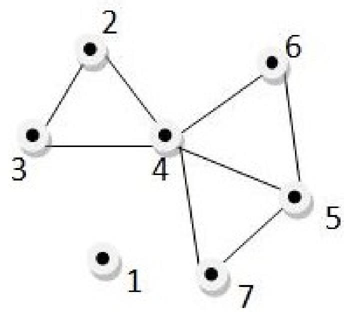

FIGURE 3.1 – An example of a 7-person trust relationship network

According to the group approach, 5 should be excluded from the groups S_3 and S_4. Thus, 4 and 7 also sever their links with 5. Therefore, the coalition structure that will form if the deviation occurs is $\gamma' \equiv \{S_1, S_2, S_3^1, S_4^1, S_5\}$ where $S_3^1 = \{4, 6\}$, $S_4^1 = \{4, 7\}$, and $S_5 = \{5\}$.

According to the network approach (level-1 punishment), only individual 6 will sever her link with 5. Therefore, the coalition structure that will form if the deviation occurs is $\gamma'' \equiv \{S_1, S_2, S_3^1, S_4\}$ where $S_3^1 = \{4, 6\}$.

In real life, each individual who is linked to 5 and is aware of 5's decision to deviate, takes strategically her own decision to sever the link with 5 or not. It can be optimal for 4 or 7 to keep the link with 5. For example, if 4 is highly pessimistic about the future, she may decide to sever the link with 5. In contrary, if 7 is highly optimistic, she may decide to keep the link with 5. This decision may also depend on the relative wealth of 5. This strategic decision is the idea behind the *endogenous punishment* that we introduce in this chapter.

We proceed by defining the endogenous punishment and we show that this punishment scheme can replicate, the weak, the strong and even the level-q punishment schemes. We show that the decision to sever a link with a deviator depends indeed on the wealth of this deviator and also on the perception of future by the person who is aware of a deviation. We characterize endogenously stable overlapping coalition structures as a result of two effects : the gain in terms of consumption and the loss in terms of links. We also derive comparative statics for consumption under two well-known sharing norms.

The chapter is organized as follows. In Section 3.2, we develop the settings of the multilateral insurance. In Section 3.3, we characterize endogenously stable informal insurance groups and we conclude in Section 3.4.

3.2 Networks and transfer norms

3.2.1 Endowment

Consider a population N of n individuals. At any date, a state of the nature $\theta \in \Theta$, Θ finite, is realized with a probability $p(\theta)$. As a result, there obtains a vector of income realization $y(\theta) = (y^i(\theta))_{i \in N}$. We assume that given $\theta \in \Theta$, realized incomes $y^i(\theta)$ are positive and not perfectly correlated [7] across individuals. [8]

In the remainder, if there is no confusion, we will use variables without their argument θ.

3.2.2 Network based group formation

To set the formation of risk sharing groups, we need some definitions.

Let $N = \{1, 2, \ldots, n\}$ be the set of all individuals. A *coalition* S is a non-empty subset of N. A cover γ of N is a collection of coalitions, $S_1, S_2, \ldots, S_m$, such that $\bigcup_{k=1}^{m} S_k = N$: thus, $\gamma \equiv \{S_1, S_2, \ldots, S_m\}$. We use the expression *overlapping coalition structure* to designate a cover whenever we want to address its structure.

A network g is a list of unordered pairs of individuals $(i, j) \in N \times N$, ij for simplicity, and $i \neq j$. Let G be the set of all the networks g.

For all $g \in G$, define :

7. Not perfectly correlated is used here in the sense that knowing the realized income of one individual is not sufficient to predict the one of others. This is a key difference between this chapter and the following chapter because in this setting, the set of accessible endowments is larger.

8. It is sufficient for outputs to be not perfectly correlated to have some insurance. Otherwise, no insurance can take place.

- $N(g) \equiv \{i \in N \mid \ \exists j \in N, ij \in g\}$
- $D(g)$ as the set of all directly connected elements of $N(g)$. Formally, $D(g) \equiv \{S \subseteq N(g) \mid \ \forall i, j \in S, ij \in g\}$.
- $\mathcal{C}l(g) \equiv \{S \in D(g) \mid \ \nexists S' \in D(g) \ s.t. \ S \subset S'\}$ as the set of cliques (completely directly connected elements) of $N(g)$.
- $I(g) \equiv N \setminus N(g)$ as the set of all singletons that have no link with any other members according to the network g.
- a component C as a set of completely connected elements of $N(g)$.
- a subnetwork of g as a network of N with at most all the links of G.

Definition 15. *The set $\gamma_g \equiv I(g) \cup \mathcal{C}l(g)$ is the cover representation of g.*

Remark 14. For all $g, g' \in G$, $g \neq g' \Rightarrow \gamma_g \neq \gamma_{g'}$ (Agbaglah, 2011). Therefore, γ_g can be used to represent a network g. This structure is useful in our setting because it suggests how coalitions can be formed based upon direct links inherited from the preexisting trust relationship network.

We use the overlapping coalition structure to account for the following empirical findings. Informal insurance groups are homogenous groups (Grimard, 1997, with data from Côte d'Ivoire) and they are overlapping (De Weerdt and Dercon, 2005, with data from Tanzania). These groups are formed given an important requirement : trust (Fafchamps and Lund, 2003, with data from Philippines). Trust is fundamental before any insurance arrangement. Only individuals who trust each other can build reciprocal transfers agreements. The reason is simply that, because of the absence of formal institutions, trust reduces information cost and solves enforceability problems.

3.2.3 Multilateral norms

In a population N, individuals are directly linked by trust relationships. These links induce a trust relationship network g on N. Therefore we use the overlapping coalition structure γ_g to represent the repartition of coalitions as risk sharing groups where insurance negotiations commence.

A multilateral norm is a specification of individuals, transfers, incomes, and consumptions for all the individuals in each coalition $S \in \gamma_g$.

When $\theta \in \Theta$ is drawn and $y^i(\theta)$ is known for each $i \in N$, each individual dispatches her realized income into coalitions she belongs to. For each individual $i \in N$, we denote by $y^i_{\gamma_g}$ the collection of her allocated incomes. Thus, $y^i_{\gamma_g} = (y^i_S)_{S\in\gamma_g}$ where y^i_S is the nonnegative portion of y^i that individual i *allocates* to the coalition S. Notice that, $y^i_S = 0$ if $i \notin S$ and $y^i_S > 0$ if $i \in S$. Therefore, we have $y_i = \sum_{S\in\gamma_g} y^i_S$. The intuition is that if an individual does not belong to a coalition, she does not contribute to it. But, if she belongs to several coalitions, her income will be allocated to all of them according to some rule (either a personal rule or a common rule). We are interested in the rules that guide this allocation. If the rule is individual and strategic, then it may be the consequence of some utility maximization problem. It may be the allocation that maximizes risk sharing. We do not intend to follow that pattern in this chapter. Instead, we follow Fafchamps and Gubert (2006, with Philippines data) who find no evidence for networks built to maximize risk sharing. Therefore, we state that allocation rules are common. These rules are secular norms that guide income allocation within coalitions.

One way to think of the norm is to state that each individual i is characterized by a vector of weights $\beta^i_{\gamma_g} = (\beta^i_S)_{S\in\gamma_g}$ such that $\sum_{i\in S\in\gamma_g} \beta^i_S = 1$ and $y^i_S = \beta^i_S y^i$. The weight, β^i_S is nothing but the fraction of her income that individual i allocates to the coalition S. Since we state that the allocation rules are common rules, we take the weights exogenous to our model. Notice that if i belongs to only one coalition S, then $\beta^i_S = 1$ and $\beta^i_{S'} = 0$ for $S' \neq S$. Hence if γ_g is a partition, then $\beta^i_S = 1$ for all i and S and all $S \in \gamma_g$.

Let $c^i_S(\theta)$ (c^i_S if no confusion) denote the fraction of individual i's consumption, that comes from the coalition S, and $z^i_S(\theta)$ (z^i_S if no confusion) the net transfer from i within the coalition S.

Thus, given a coalition S, a multilateral norm is a mapping $\mathcal{M}$, such that $(z^i_S)_{i\in S} = \mathcal{M}\big(S, (y^i_S)_{i\in S}\big)$ subject to the following constraint $y^i_S - z^i_S = c^i_S$. The overall consumption for an individual $i \in N$ is $c^i = \sum_{i\in S\in\gamma_g} c^i_S$ which is the summation of all coalitional consumptions from the whole informal insurance organization.
The research question that we investigate here is : *how can we characterize overlapping coalition structures where every individual abide by the norms* ? Such coalition structure is denoted *stable*.

3.2.4 Transfers

We model informal insurance as arrangements between homogenous groups of individuals. Transfers occur only between directly linked individuals. If a state of nature is realized, and the consumption of each individual in a coalition is determined according to the norm, the overall transfer of each individual is known. However, there are multiple ways to specify bilateral transfers. Here we propose the proportional transfer based on empirical findings. Three possible situations arise within a coalition.

– If an individual i has an amount of allocated income equal to the consumption designated by the norm, then i keeps her income. Her net transfer is null.
– If an individual i has an amount of allocated income greater than the consumption designated by the norm, then i shares her excess of income proportionally with individuals who gain less than the designated amount of consumption. Her net transfer is positive and equals the amount of excessive consumption good.
– If an individual i has an allocated income less than the consumption designated by the norm, then i receives her lack of income proportionally from individuals who gain more than the designated amount of consumption. Her net transfer is negative and equals the amount of lack of consumption good.

For illustration, we consider the following 3-person network. In brackets are the incomes.

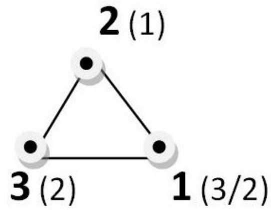

FIGURE 3.2 – An example of a 3-person network with realized incomes

If for example the multilateral sharing norm is equal sharing, the total income (4.5) is equally shared and individual consumption is equally fixed to 1.5. To obtain this consumption distribution, 1 has to make a net transfer of 0 ; 2 a net transfer of -0.5 ; and 3 a net transfer of 0.5.

Definition 16. *Consider a coalition $S \in \gamma_g$, a realized income y, and a multilateral norm $\mathcal{M}$ such that $(z_S^i)_{i\in S} = \mathcal{M}\big(S, (y_S^i)_{i\in S}\big)$. Let E_S^i denote the "excess" of consumption for individuals $i \in S$ defined by $E_S^i \equiv y_S^i - c_S^i$. The proportional transfer scheme is such that :*

- *If $E_S^i = 0$, then $z_S^i = 0$ and $z_S^{il} = 0$ for all $l \in S \setminus \{i\}$.*
- *If $E_S^i > 0$, then $z_S^i = \sum_{\{l/E_S^l<0\}} z_S^{il}$ (where z_S^{il} is the transfer by i to l in the coalition S) with $z_S^{il} = \frac{E_S^l}{\sum_{\{k/E_S^k<0\}} E_S^k} E_S^i$, and $z_S^{il} = 0$ for all $l \in S$ with $E_S^l \geq 0$.*
- *If $E_S^i < 0$, then $z_S^i = \sum_{\{l/E_S^l>0\}} z_S^{il}$, with $z_S^{il} = -z_S^{li}$.*

If $i \notin S$, then $z_S^i = 0$.[9]

3.2.5 Externalities

Valuable people in agrarian economies are not only the wealthiest, but also people with high social privileges (number of friends, respect, power in common decisions, favor received and so on).[10] These social privileges may not be convertible into

9. This last part of the definition formally states that there is no transfer between disjoint coalitions

10. Sometimes wealthiest people are also high social privilege ones.

consumption good. In our setting, we characterize social privileges by the amount of links in the trust relationship network. Notice that the network exists prior to insurance arrangements but trough social privileges, it affects the wellbeing of individuals. For this reason, we address this social privileges as *externalities.* These externalities can be positive or negative. When individuals with a big amount of links (in the trust relationship network) have a bad harvest, they will benefit from more favor from their linked partners : that pattern exhibits positive externalities. In contrast, if time is more valuable and that one needs to spend an amount of time with each of her linked partners, a big amount of link is more costly : that pattern exhibits negative externalities.

Formally, we endow each individual with an utility function that depends not only on the consumption good, but also on the architecture of the trust relationship network.

Definition 17. *Let g be a network on N, and let g' be a subnetwork of g.*
The utility function exhibits ***positive externalities*** *if $U(c_i, g) \geq U(c_i, g')$ and the inequality is strict for at least one individual.*
The utility function exhibits ***negative externalities*** *if $U(c_i, g) \leq U(c_i, g')$ and the inequality is strict for at least one individual.*
The utility function exhibits ***no externalities*** *if $U(c_i, g) = U(c_i, g')$ for all g' and all i.*

Following Ambrus et al. (2010), we state that these externalities arise not at the village level, but at the level of homogenous islands. At the coalition level, say *local externalities* and at the component level, say *global externalities.* We borrow here from Ambrus et al. (2010), the fact that the external effect is a substitute to the consumption good. But, the difference resides in the fact that we do not allow here external effect to be necessary convertible into consumption. We postulate instead an additive separable functional form. Thus, $U(c, g) = u(c) + f(g)$ where u is a smooth function in consumption, strictly concave and increasing. The functional f is a smooth function, increasing (respectively decreasing) in the *number of directs links* of i in the case of positive (respectively negative) *local* externalities, and increasing (respectively decreasing) in the *number of links* (either directs or indirects) of i in the case of positive (respectively negative) *global* externalities. If there are no externalities, then $f \equiv 0$. Furthermore, we set that $f(\emptyset) = 0$. An isolated individual (singleton) is not subject to any external effect.

3.2.6 Enforcement constraints

This chapter is all about **informal** insurance. This is a reciprocal transfer agreement that is not formally regulated : no collateral is required, no legal court to complain, and no signed paper as testimony. In some situations, individuals may find it suboptimal to comply. Therefore, to survive, such agreement needs to be self-enforcing in order to discourage opportunist behaviors. One who reneges on her prescribed duty needs to be "punished". If the literature is unanimous on such punishments, its severity depends on the approach. According to the group approach, in case of default, the guilty individual will be pushed out of the group. All the links with her will be severed. This is quite a strong punishment. According to the network approach, only direct victims will sever their links with the deviator. This is quite a weak punishment. The range between the weak and the strong punishment is due to the modeling of the reaction of *third parties*. These are individuals who are not directly victims of the deviation, but are linked to the victim and therefore aware of this deviation. As an alternative to these extreme punishments, an intermediary case is developed by Bloch, Genicot and Ray (2008), the level-q punishment. According to this punishment, *following a deviation, all agents who are connected to a victim by a path not exceeding length q (but not via deviator) sever direct links (if any) with the deviator*. This intermediary punishment, though it is very interesting, presents two limits : q is exogenous, and is the same for all the individuals.

In this chapter we take an eclectic view by designing a punishment scheme that reflects reality. In real life what happens if an individual in a population reneges on her duty ? Surely, the direct victims will sever their links with the deviator. However, the reaction of the third parties is far from being obvious. Their decision is individual and strategic. All individuals who are informed about a deviation become more cautious. Nevertheless, their reaction follow a cost-benefit computation about keeping or severing the link with the deviator. The reason is that it is not granted that a deviating partner will do the same thing to them in the future.

Formally, suppose that each individual can still trust (but only up to some level) her linked partner who comes to deviate to another one. Depending of her type or the history, she can decide if the current deviator remains reliable or not. Since we restrict ourselves to stationary behaviors, we will focus on type alone to define our punishment scheme. An individual i is of type $t^i \in [0,1]$ if she believes with probability t^i that a current deviator will deviate to her also in the future. The type t^i is nothing but the trust probability over future reciprocity. For example, if all individuals are of type $t^i = 1$, then the population is pessimistic and the strong punishment obtains. If all individuals are of type $t^i = 0$, then the population is optimistic and the weak punishment obtains.

Deviation

If it happens that according to the agreement an individual i should make transfers but she reneges on her duty, we say that i *deviates*. Each individual i can only deviate to directly linked partners. Formally, each individual i can only deviate to individuals in a set $D \subseteq \left(\cup_{i\in S\in\gamma_g}\{S\}\right)\setminus\{i\}$ from the coalitions she belongs to. Given a network $g \in G$, a deviation will induce some further deviations up to some "stable" network g' (g' can be the empty network). In case of multiplicity, take g' as the one which generates the most positive (least negative) external effect. We denote by c^i_D individual i's current consumption if she deviates to others in the set D. Thus,

$c^i_D \equiv c^i + \sum_{l \in D} \sum_{S \in \gamma_g : i, l \in S} z^{il}_S$. From the next period on, i will face two types of punishment, one *direct* and the other one *indirect*. The direct punishment is the severance of the links by all individuals in D (direct victims of deviation). The indirect punishment is the severance of the links by third parties. The link severance will induce a residual network $g' \subset g$, and a residual consumption c^i_R.[11] Thus, i decides to deviate if the gain from the pair (c^i, g) is less than the gain from the pair (c^i_D, g').

Endogenous punishment

The behavior of third parties determines the severity of the punishment (weak or strong). In the case of weak punishment, third parties are passive. In contrast, they are active in case of strong punishment since they also punish the deviator (if they are linked to her). In this chapter we explore another view : *why if third parties behave strategically.* A third party individual severs the link with a deviator if and only if it worth less for her to keep this link than to sever it : this is the *endogenous punishment.* Suppose that an individual j is a deviator and i a third party. Before deciding to sever her link with j, a third party i, cautious of future possible deviations by j, compares her per period payoff from link severance denoted V^i_{Sj} to the one denoted V^i_{Kj} from link keeping. Information is complete so everybody in the same component[12] knows the residual network following each action (severing the link with the deviator, g_{Sj}, or keeping it, g_{Kj}).

Definition 18.

Suppose that an individual j deviates. The endogenous punishment scheme consists of a series of simultaneous actions :

- *If an individual i is a direct victim of the deviation, then she severs her link with j.*

11. We use the letters D for default and R for reaction.
12. Empirical papers show that information is shared in islands of individuals represented here by components.

– *If an individual i is a third party of the deviation, i severs her link with j if and only if* $V^i_{Kj} \leq V^i_{Sj}$

We already show that the endogenous punishment replicates the strong and the weak punishments. What about the level-q punishment ?

Proposition 8.

Let q be a an integer. Suppose that individual i's type is such that :

$$\begin{cases} \text{If the lengh of the path connecting } i \text{ to a direct victim is not greater than } q, t^i = 0 \\ \text{Otherwise } , t^i = 1 \end{cases}$$

then the endogenous punishment is the level-q punishment.

The proof is straightforward since the link severance from the endogenous punishment is equivalent to the one from the level-q punishment.

Lemma 9. *A third party i severs her link with a deviator j if and only if*

$$\sum_\theta p(\theta)[t^i U(c^i_{Kj}, g_{Kj}) + (1-t^i)U(c^i_{Sj}, g_{Kj})] \leq \sum_\theta p(\theta)U(c^i_{Sj}, g_{Sj}),$$

where $c^i_{Kj}(\theta)$ *is the amount of consumption for i if she keeps the link with j, and* $c^i_{Sj}(\theta)$*, the amount of consumption for i if she severs the link with j.*

Proof.

Notice that, V^i_{Sj} is the expected per period utility of severing the link with j, and V^i_{Kj} is the expected per period utility of keeping this link. Thus, $V^i_{Sj} \equiv \sum_\theta p(\theta)U(c^i_{Sj}, g_{Sj})$, and $V^i_{Kj} \equiv \sum_\theta p(\theta)[t^i U(c^i_{Kj}, g_{Kj}) + (1-t^i)U(c^i_{Sj}, g_{Kj})]$; and a third party i severs her link with a deviating j if and only if $V^i_{Sj} \geq V^i_{Kj}$. □

Notice that in the LHS of the inequality, we have g_{Kj} which is the subnetwork obtained if i decides to keep j. At the period of decision (harvest for example), if i decides to keep j, the external effects take place up to the next period.

For an individual third party i, the indirect punishment against the deviator j follows a rational computation. There is a positive effect to keep the deviator, especially if she is wealthy. But the consequence can be dramatic if this deviator comes to renege on her transfer to i. For this reason, each individual third party i evaluates the relative value of the indirect external loss and the direct consumption coverage loss following individual j's deviation to a third party i. To evaluate this relative value, let $k_j^i \equiv \frac{f(g_{Sj})-f(g_{Kj})}{\sum_\theta p(\theta)\{u(c^i_{Kj})-u(c^i_{Sj})\}}$. Each individual third party i's evaluation of this value determines her decision to sever a link or not. Notice that k_j^i is endogenous and results from a cost-benefit analysis.

Proposition 9. *For an individual third party i and a deviator j,*

- *If the deviator j's icome is high enough then :*
  - *If there are no externalities, or in case of positive externalities, then i* ***never severs*** *the link with j. This decision is independent of t^i.*
  - *If there are negative externalities, then i severs the link with j only if $k_j^i \geq t^i$.*
- *If the deviator j's income is low enough then :*
  - *If there are no externalities, or in case of negative externalities, then i* ***always severs*** *the link with j.*
  - *If there are positive externalities, then i severs the link with j if and only if $k_j^i \leq t^i$.*

Remark 15.

- First, if the number of links increases social privileges (positive externalities), nobody severs a link with a wealthy deviator. For a poor deviator, her links are needed but she is a burden in the sense that she is poor. Therefore only poor deviators with low relative external contribution are severed.

- Second, if the number of links decreases social privileges (negative externalities), the link with a poor deviator is systematically severed. however, if a rich deviates, her income is needed but she is a burden in terms of link. Therefore, link severance occurs if and only the loss in terms of social privilege is relatively high.

- It is crucial to notice that the threshold for the decision to sever or not a link is the trust probability t^i. The confrontation between an exogenous parameter t^i and an endogenous one k^i_j determines the decision to keep a deviator or not.

Proof.

$V^i_{Kj} \leq V^i_{Sj} \Leftrightarrow \sum_\theta p(\theta)\{[t^iU(c^i_{Kj}, g_{Kj}) + (1-t^i)U(c^i_{Sj}, g_{Kj})] - U(c^i_{Sj}, g_{Sj})\} \leq 0$

$\Leftrightarrow \sum_\theta p(\theta)\{[t^iu(c^i_{Kj}) + t^if(g_{Kj}) + (1-t^i)u(c^i_{Sj}) + (1-t^i)f(g_{Kj})] - [u(c^i_{Sj}) + f(g_{Sj})]\} \leq 0$

$\Leftrightarrow t^i \sum_\theta p(\theta)[u(c^i_{Kj}) - u(c^i_{Sj})] \leq f(g_{Sj}) - f(g_{Kj})$ because g_{Sj}, g_{Kj}, t^i are independent of θ

Case 1 : No externalities

In case of no externalities, $f(g_{Sj}) - f(g_{Kj}) = 0$

Hence, $V^i_{Kj} \leq V^i_{Sj} \Leftrightarrow \sum_\theta p(\theta)[u(c^i_{Kj}) - u(c^i_{Sj})] \leq 0$

Case 2 : Positive externalities

In case of positive externalities, $f(g_{Sj}) - f(g_{Kj}) \leq 0$

Hence, $V^i_{Kj} \leq V^i_{Sj} \Leftrightarrow t^i \sum_\theta p(\theta)[u(c^i_{Kj}) - u(c^i_{Sj})] \leq f(g_{Sj}) - f(g_{Kj}) \leq 0$

Case 3 : Negative externalities

In case of negative externalities, $f(g_{Sj}) - f(g_{Kj}) \geq 0$

Hence, $V^i_{Kj} \leq V^i_{Sj} \Leftrightarrow t^i \sum_\theta p(\theta)[u(c^i_{Kj}) - u(c^i_{Sj})] \leq f(g_{Sj}) - f(g_{Kj}) \geq 0$

Furthermore, if j is more wealthy than others whatever the realized state is, then keeping j is more worthy, in terms of consumption, than severing the link with her. Therefore,$u(c^i_{Kj}) - u(c^i_{Sj}) > 0$. On the other hand, if j is poor enough, $u(c^i_{Kj}) - u(c^i_{Sj}) < 0$. The results of the proposition obtain from this observation and the three cases. □

The proposition shows that the decision to sever a link depends indeed on the income of the deviator. If the deviator is rich enough, one can still keep the link with her, even with the fear that she can deviate in the future. These results match perfectly what we observe in real life. *It is easier to forgive the rich.* The fact that we obtain this result with a very general setting (not specifying neither a functional form for the utility function (over consumption), nor a functional expression of the externalities function) shows that our model behaves well.

3.3 Endogenous stability

In the sequel, we consider stationary strategies only. These are strategies such that the decision to keep or sever a link by a third party or the decision to deviate by an individual does not depend on the history. Following a deviation, at least one link is severed. Therefore, we define recursively *endogenous stability* on the number of links.

• For an isolated individual (no link), $V^{*i}(\emptyset) \equiv \sum p(\theta)U(y^i(\theta), \emptyset)$. This is the average per period expected utility, normalized by $(1-\delta)$. Obviously, a component of size one is stable.

• Recursively, suppose that the endogenous stability is defined for any network g', subnetwork of some network g of m individuals, $m = 1, \ldots, n-1$.

• Consider a network g of n individuals, a consumption allocation c^i for i, and a vector of per period expected payoffs V^i . Consider a realization θ of the state of nature. We have two possible situations :

- If there is no deviation, an individual i's expected utility normalized by $(1-\delta)$ is $(1-\delta)U(c^i(\theta), g) + \delta V^i$.

- If i deviates to individuals in a set D, her expected utility utility is $(1-\delta)U(c^i_D(\theta), g) + \delta V^i_R(g')$, where g' is the resulting residual stable subnetwork of g obtained from the severance of links by individuals in R who react to the defection of i.

Thus, an individual i deviates if and only if $(1-\delta)U(c^i(\theta), g) + \delta V^i(g) \leq (1-\delta)U(c^i_D(\theta), g) + \delta V^i_R(g')$. If there are multiple such g', the inequality should be verified for all.

Definition 19. *An overlapping coalition structure γ_g is endogenously stable if and only if*

$$(1-\delta)U(c^i(\theta_0), g) + \delta V^i(g) > (1-\delta)U(c^i_D(\theta_0), g) + \delta V^i_R(g')$$

for all $D \subseteq \left(\cup_{i \in S \in \gamma_g} \{S\}\right) \setminus \{i\}$, for all $i \in N$, and for all realized states $\theta_0 \in \Theta$.

From this definition, the following characterization of endogenous stability obtains.

Lemma 10. *An overlapping coalition structure γ_g is endogenously stable if and only if*

$$U(c^i_D(\theta), g) - U(c^i(\theta), g) < \frac{\delta}{(1-\delta)}[V^i(g) - V^i_R(g')]$$

for all $D \subseteq \left(\cup_{i \in S \in \gamma_g} \{S\}\right) \setminus \{i\}$, for all $i \in N$, and for all realized states $\theta_0 \in \Theta$.

Remark 16. The Lemma is a direct consequence of the definition. The gain of cheating today is less than the actual value of the punishment from the next day on. The same result is obtained in different contexts by Coate and Ravallion (1993), and Bloch, Genicot and Ray (2008). The difference resides in the construction of V^i_R that depends deeply on the punishment scheme. After all, an insurance organization is endogenously stable if and only if respecting the agreement is more worthy than cheating.

3.3.1 Characterization of endogenous stability

For each individual $i \in N$, let $\mathcal{S}_i \equiv \left(\bigcup_{S \in \gamma_g : i \in S} S\right) \setminus \{i\}$ denote the set of all individuals linked to i. Let c^i_D denote i's current consumption if she deviates to a set $D \subseteq \mathcal{S}_i$. Let c^i_R denote i's consumption from the next day on, due to the residual stable network g_R following the endogenous punishment. Suppose that we recursively define stability for networks of less than n individuals.

Theorem 7. *An overlapping coalition structure γ_g is endogenously stable if and only if for all individual i, for all set $D \subseteq \mathcal{S}_i$, and for all realized states $\theta_0 \in \Theta$,*

$$u(c_D^i(\theta_0)) - u(c^i(\theta_0)) < \frac{\delta}{1-\delta}\Big\{\sum_{\theta\in\Theta} p(\theta)\big(u(c^i(\theta)) - u(c_R^i(\theta))\big) + f(g) - f(g_R)\Big\}.$$

Interpretation.

This theorem shows the existence of two effects. The first effect is direct and consists of the lack of consumption smoothing coverage following a punishment. The second effect is indirect and concerns external losses following links severance. Thus, an overlapping coalition structure is endogenously stable if and only if the gain from defection today is less than the actual value of the resultant of the direct and the indirect effects.

Proof.

If an individual i deviates to a set D, her current consumption is $c_D^i = c^i + \sum_{l\in D}\sum_{S\in\gamma_g:i,l\in S} z_S^{il}$.
Stability of $\gamma_g \Leftrightarrow \forall\theta_0\in\Theta, \forall i\in N, \forall D\subseteq\mathcal{S}_i, U(c_D^i(\theta_0),g) - U(c^i(\theta_0),g) < \frac{\delta}{(1-\delta)}[V^i(g) - V_R^i(g_R)]$.
From :

1. $U(c_D^i(\theta_0),g) - U(c^i(\theta_0),g) = u(c_D^i(\theta_0)) + f(g) - u(c^i(\theta_0)) - f(g)$
2. $V^i(g) = \sum_{\theta\in\Theta} p(\theta)\big(u(c^i(\theta)) + f(g)\big) = \sum_{\theta\in\Theta} p(\theta)u(c^i(\theta)) + f(g)$ because $f(g)$ is independent from θ and $\sum_{\theta\in\Theta} p(\theta) = 1$
3. $V_R^i(g_R) = \sum_{\theta\in\Theta} p(\theta)\big(u(c_R^i(\theta)) + f(g_R)\big) = \sum_{\theta\in\Theta} p(\theta)u(c_R^i(\theta)) + f(g_R)$ because $f(g_R)$ is independent from θ and $\sum_{\theta\in\Theta} p(\theta) = 1$

we have the following :
Stability of $\gamma_g \Leftrightarrow \forall\theta_0\in\Theta, \forall i\in N, \forall D\subseteq\mathcal{S}_i, u(c_D^i(\theta_0)) - u(c^i(\theta_0)) < \frac{\delta}{1-\delta}\{\sum_\theta p(\theta)\big(u(c^i(\theta)) - u(c_R^i(\theta))\big) + f(g) - f(g_R)\}$. □

It is granted in the literature that only an individual with high realized income can renege on her transfer. But we observe also in real life that some situations may force an individual with low realized income (with negative excess of consumption as in Definition 17) to refuse a transfer. The following result investigates these situations.

Corollary 1. *With negative externalities, an individual may deviate with a low income realization. She never deviates in other cases.*

Interpretation.

All the economics literature on this matter claims intuitively that deviation occurs only with high realized income. For example Genicot and Ray (2002) assume with a model without externalities that deviation occurs only if the deviator has a higher realized income. We confirm this assumption here by proving it formally and isolating situations where this assumption may not hold.

Proof.

If i has a low realized income, $\sum_{l\in D}\sum_{S\in\gamma_g:i,l\in S} z_S^{il} < 0$, thus $u(c_D^i(\theta_0)) - u(c^i(\theta_0)) < 0$, and we use the theorem to conclude. □

The following result investigates the relation between stability and discount factor.

Corollary 2. *In the presence of positive externalities and in case of no externalities, all overlapping coalition structures are endogenously stable for sufficiently patient individuals. However, in case of negative externalities, an overlapping coalition structure is endogenously stable for sufficiently patient individuals if and only if the direct effect of deviation dominates the indirect effect.*

Proof.

For sufficiently patient individuals, δ tends to 1 and $\frac{\delta}{(1-\delta)}$ to infinity. Also, $\sum_\theta p(\theta)\big(u(c^i(\theta))-u(c^i_R(\theta))\big) \geq 0$ We have the following cases :

1. No externalities : $f(g) - f(g_R) = 0$;
2. Positive externalities $f(g) - f(g_R) \geq 0$
3. Negative externalities $f(g) - f(g_R) \leq 0$

Hence we have the sign of the RHS of the inequality in Theorem 7 and the result of the corollary is straightforward. □

What happens to insurance in communities where individuals are heterogenous in terms of income ? We think of situations where some individuals have access to higher income compared to others ? Does stability hold ?

Corollary 3. *Suppose there exists at least one individual wealthier than others (independently of the realized state). In case of no externalities or positive externalities, an overlapping coalition structure γ_g is endogenously stable if and only if individuals are patient enough.*

Interpretation.

With high discounting, there can exist endogenously stable informal insurance groups with heterogeneity in income. But if we consider non-extreme discounting, endogenously stable informal insurance groups exhibit homogeneity in income. These results reinforce empirical finding of islands of homogenous individuals. They suggest to investigate homogenous group of individuals who have access to the same set of income.

Proof.

It is sufficient to notice that $u(c^i_D(\theta_0))-u(c^i(\theta_0)) > 0$ for the very rich individual for all θ_0. Then we obtain from Theorem 7 that $\delta > \frac{u(c^i_D(\theta_0))-u(c^i(\theta_0))}{u(c^i_D(\theta_0))-u(c^i(\theta_0))+\sum_\theta p(\theta)\Big(u(c^i(\theta))-u(c^i_R(\theta))\Big)+f(g)-f(g_R)}$

where the RHS is in $(0,1)$ because $f(g)-f(g_R)\geq 0$. □

3.3.2 Endogenous stability and sharing norms

For expositional perspective, we consider two distinct sharing norms : the *equal sharing norm* and the *norm with private domain.*[13] According to the equal sharing norm, individuals in the same coalition put their allocated incomes together and share the total equally among themselves. According to the norm with private domain, each individual in a coalition keeps an incompressible portion of her allocated income and all of them put the remaining together and share it within the coalition.

Definition 20. *A multilateral norm $\mathcal{M}$ on a coalition $S \in \gamma_g$ of s individuals is the **equal sharing norm** if according to $\mathcal{M}$, each individual $i \in S$ receives an amount c^i_S of consumption good such that $c^i_S = \frac{1}{s}\sum_{l\in S} y^l_S$.*
*A multilateral norm $\mathcal{M}$ on a coalition $S \in \gamma_g$ of s individuals is a **norm with private domain** if according to $\mathcal{M}$, each individual $i \in S$, receives an amount c^i_S of consumption good, such that $c^i_S = \alpha^i y^i_S + \frac{1}{s}\sum_{l\in S}(1-\alpha^l)y^l_S$, $\alpha^i \in (0,1)$.*

In the sequel, we provide firstly results for equal sharing norm as a benchmark. Thereafter, we investigate a norm with private domain.
Consider an income realization and an individual i contemplating a deviation. Let $\overline{n}$ be the number of individuals with lower than the prescribed (by the norm) allocated income, and $\overline{p}$ the number of individuals with higher than the prescribed (by the norm) allocated income.

Theorem 8. ***Equal sharing norm***
If an individual i contemplates deviating to a set D, then her current consumption c^i_D is such that :

- *in case of positive externalities, c^i_D increases with y^i and decreases with $\overline{n}$ and the size of D,*

13. For details about sharing norms, see Genicot and Ray (2002).

- *in case of negative externalities,* c^i_D *increases with* y^i *and with* $\overline{p}$ *and decreases with the size of* D*.*

Interpretation.

In case of positive externalities, the incentive to deviate increases with the level of realized income. If there are rich individuals, it is more likely to see them deviate. If deviation occurs, the best thing to do is to refuse transfers to as many as possible individuals in N. If a very few number of individuals have a negative shock in their incomes, the incentive to deviate increases.

In case of negative externalities, the incentive to deviate increases with the level of realized income. If there are rich individuals, then it is more likely to see them deviate. In case of deviation, the best thing to do is to refuse transfers to as few as possible individuals in P. If a very few number of individuals have a positive shock in their incomes, the incentive to deviate diminishes.

Proof.

Case1 : Consider an individual i who belongs to only one coalition S of size s in a stable informal insurance structure γ_g. Consider that a state θ_0 is realized, with the income distribution in S, $(y^i_S)_{i\in S}$. According to the equal sharing norm, i consumes $c^i(\theta_0) = \frac{1}{s}\sum_{l\in S} y^l_S$. Let $\mathcal{N} \equiv \{l \in S : E^l_S < 0\}$, and $\mathcal{P} \equiv \{l \in S : E^l_S > 0\}$ the sets of respective sizes n and p.

Subcase1 : Consider an individual i such that $i \in P$ then i makes positive transfers to other individuals in N. Thus if she decides to default, she can refuse transfers to the individual set $D \subseteq N$, of size $d \leq n$.

$$\begin{aligned}
c_D^i &= c^i + \sum_{l \in D} z_{Sl}^i = \frac{1}{s}\sum_{l \in S} y_S^l + \frac{1}{\sum_{l \in N} E_N^l}\sum_{l \in D} E_S^l E_S^i \\
&= \frac{1}{s}\sum_{l \in S} \beta_S^l y^l + \frac{1}{\sum_{l \in N} \beta_S^l y^l - \frac{n}{s}\sum_{l \in S} \beta_S^l y^l}\left(\sum_{l \in D} \beta_S^l y^l - \frac{d}{s}\sum_{l \in S} \beta_S^l y^l\right)\left(y^i - \frac{1}{s}\sum_{l \in S} \beta_S^l y^l\right)
\end{aligned}$$

Hence $c_D^i(\theta_0) = f(n(\theta_0), d, y^i(\theta_0))$ where : $\frac{\partial f}{\partial n} < 0; \frac{\partial f}{\partial d} < 0; \frac{\partial f}{\partial y^i} > 0$

We have $f(n, d, y^i) = a + \frac{1}{b(n) - na}(c(d) - da)(y^i - a);$ with $b(n) - na < 0; c(d) - da < 0; y^i - a > 0$

$\frac{\partial f}{\partial n} = -\frac{c(d)-da)(y^i-a)}{(b(n)-na)^2}(b'(n) - a)$ $b'(n)$ is the marginal aggregate consumption good for an individual in N ; it is the less than a. So $b'(n) - a < 0 \Rightarrow \frac{\partial f}{\partial n} < 0$

$\frac{\partial f}{\partial d} = \frac{(y^i-a)}{b(n)-na}(c'(d) - a)$ $c'(d)$ is the marginal aggregate consumption good for an individual in D ; it is the less than a. So $c'(d) - a < 0 \Rightarrow \frac{\partial f}{\partial d} > 0$

$\frac{\partial f}{\partial y^i} = \frac{(c(d)-da)}{b(n)-na} > 0$

$R = D \cup T$, where T is the set of third parties : $T = \{l \in S : t^l \sum_\theta p(\theta)[u(c_{Ki}^l) - u(c_{Si}^l)] \leq f(g_{Si}) - f(g_{Ki})\}$

Subsubcase 1 : l belongs to S only.

$c_{Si}^{0l} = \frac{1}{s-1}\sum_{k \in S-\{i\}} \beta_S'^k y^k$ where $\beta_S'^k$ is the new repartition of income

$c_{Ki}^{0l} = \frac{1}{s-1}\sum_{k \in S-\{i\}} \beta_S'^k y^k + \frac{1}{s-r+1}\sum_{k \in (S-R)\cup\{l\}} \beta_S'^k y^k$ where $\beta_S'^k$ is the new repartition of income

Subsubcase 2 : l belongs to multiple coalitions $\{S, S_j, j\}$.

$c_{Si}^l = c_{Si}^{0l} + \sum_j c_j'^l$ where $c_j'^l$ is the new amount of consumption good from other coalitions

$c_{Ki}^l = c_{Ki}^{0l} + \sum_j c_j'^l$ where $c_j'^l$ is the new amount of consumption good from other coalitions

Subcase2 : Consider an individual i such that $i \in N$ then i makes negative transfers to (receive positive transfers from) other individuals in P. Thus if she decides to default, she can refuse transfers to the individual set $D \subseteq P$, of size $d \leq p$.

$$c_D^i = c^i + \sum_{l \in D} z_{Sl}^i \text{ with } z_{Sl}^i = -z_{Si}^l$$
$$= \frac{1}{s}\sum_{l \in S} y_S^l - \frac{1}{\sum_{l \in N} E_N^l}\sum_{l \in D} E_S^l E_S^i \text{ But } \sum_{l \in N} E_N^l = -\sum_{l \in P} E_P^l \text{ So}$$
$$c_D^i = \frac{1}{s}\sum_{l \in S} \beta_S^l y^l + \frac{1}{\sum_{l \in P} \beta_S^l y^l - \frac{P}{s}\sum_{l \in S} \beta_S^l y^l}(\sum_{l \in D} \beta_S^l y^l - \frac{d}{s}\sum_{l \in S} \beta_S^l y^l)(y^i - \frac{1}{s}\sum_{l \in S} \beta_S^l y^l)$$

Hence $c_D^i(\theta_0) = f(p(\theta_0), d, y^i(\theta_0))$ where : $\frac{\partial f}{\partial p} > 0; \frac{\partial f}{\partial d} < 0; \frac{\partial f}{\partial y^i} > 0$

We have $f(p, d, y^i) = a + \frac{1}{b(p)-pa}(c(d) - da)(y^i - a);$ with $b(p) - pa > 0; c(d) - da > 0; y^i - a < 0$

$\frac{\partial f}{\partial n} = -\frac{c(d)-da)(y^i-a)}{(b(p)-pa)^2}(b'(p) - a)$ $b'(p)$ is the marginal aggregate consumption good for an individual in P; it is the greater than a. So $b'(p) - a > 0 \Rightarrow \frac{\partial f}{\partial p} > 0$

$\frac{\partial f}{\partial d} = \frac{(y^i-a)}{b(p)-pa}(c'(d) - a)$ $c'(d)$ is the marginal aggregate consumption good for an individual in D; it is the greater than a. So $c'(d) - a > 0 \Rightarrow \frac{\partial f}{\partial d} > 0$

$\frac{\partial f}{\partial y^i} = \frac{(c(d)-da)}{b(p)-pa} > 0$

The remainder is like in the subcase1

Case2 : Consider an individual i who belongs to more than one coalition $S_j, j \in M$ of respective sizes s_j in a stable network g. Note $S = \cup_{j \in M} S_j$. Consider that a state θ_0 is realized, with the following income distribution in S, $(y_j^1(\theta_0), y_j^2(\theta_0), \ldots, y_{S_j}^{s_j}(\theta_0))$. According to the equal sharing norm, i consumes : $\sum_{j \in M} c_j^i(\theta_0)$, where $c_j^i(\theta_0) = \frac{1}{s}\sum_{l=1}^{s_j} y_j^l$.

Let $N_j \equiv \{l \in S : E_j^l < 0\}$, and $P_j = \{l \in S : E_j^l > 0\}$ the sets of respective sizes n_j and p_j, $j \in M$.

Thus, $c^i(\theta) = \sum_{j \in M} c_j^i(\theta), c_D^i(\theta) = \sum_{j \in M} c_{D_j}^i(\theta), c_R^i(\theta) = \sum_{j \in M} c_{R_j}^i(\theta)$ □

The norm with private domain prescribes that each individual i in a coalition S keeps an incompressible portion α^i of her allocated income and share the remainder equally. Thus i consumes $c^i_S = \alpha^i y^i_S + \frac{1}{s}\sum_{l\in S}(1-\alpha^l)y^l_S$, $\alpha^i \in (0,1)$ for all i.

Theorem 9. ***Norm with private domain***

If an individual i contemplates deviating to a set D, then her current consumption c^i_D is such that :

- *c^i_D increases with α^i,*
- *in case of positive externalities, c^i_D increases with y^i and decreases with $\overline{n}$ and the size of D,*
- *in case of negative externalities, c^i_D increases with y^i and with $\overline{p}$ and decreases with the size of D.*

Proof.

With the norm with private domain, $c^i_S = \alpha^i y^i_S + \frac{1}{s}\sum_{l\in S}(1-\alpha^l)y^l_S$, $\alpha^i \in (0,1)$ for all i. Thus, $c^i_S = c^i_{1S} + c^i_{2S}$, where c^i_{2S} is an equal sharing norm for $y^i_{2S} = (1-\alpha^i)y^i_S$. So the results of equal sharing norm apply to c^i_{2S}. □

3.4 Conclusion

In this chapter we introduce some important features in the economics literature on informal insurance. First, the merging of the group approach and the network approach permits to design a punishment scheme where the decision to severe a link with the deviator is strategic and therefore endogenous. Second, we account for externalities as the indirect consequence of the architecture of the trust relationship network. Third, we allow risk sharing groups to overlap as it is observed empirically. We characterize endogenously stable informal insurance groups under positive and negative externalities. After deriving general properties, we provide conditions under which deviation occurs. We show that the confrontation between the exogenous perception of the future and the endogenous relative valuation of external effects over consumption determines link severance. Finally, we derive comparative static results under two sharing norms.

Chapitre 4

Informal insurance organizations

4.1 Introduction

We investigate the viability of informal insurance arrangements across networks of individuals in a context where individual outputs are precarious. Such networks are mostly present in agrarian economies (villages in developing countries for example) [1] where individual incomes are more exposed to shocks : poor weather, illness, crop diseases, wild animals' invasion, crops damage by fire. [2] With the absence of formal credit and insurance markets, individuals cope with these idiosyncratic income shocks by developing reciprocal transfer agreements without any formal contract, signed documents, or collateral. This is known as *informal insurance.* [3]

Within the agrarian village, individuals (or households) belong to more homogenous groups (ethnic groups, families, clans, castes) [4] within which informal insurance take place. This is due to information cost and enforceability reasons. These limited groups within the village are institutions where transfers are regulated by norms. To

1. See Postner (1980) for characteristics of these economies.
2. Precarious outputs are mostly observed in rural areas but sometimes also in towns. See for instance Adaman et al. (2007) with urban Istanbul data.
3. For more information about informal insurance, see Rosenzweig (1989), Udry (1990, 1994), Townsend (1994).
4. Homogenous insurance groups are described for example by Grimarg (1997) in villages in Côte d'Ivoire.

account for that, we allow multilateral norms [5] to govern transfers in our model. Notice that if a homogenous group is composed of two individuals, the norm becomes bilateral as developed in Bloch, Genicot and Ray (2008). Theoretically, this chapter can be viewed as a generalization of bilateral norms to more than two individuals. In the remainder, we write BGR instead of Bloch, Genicot and Ray (2008). The aim of this chapter is to investigate how characteristics of these homogenous groups and multilateral norms affect the viability of informal insurance arrangements in agrarian villages.

Earlier economics literature models informal insurance organizations as "clubs" : communities (village) where individuals make reciprocal transfers in order to smooth consumption [Kimball (1988), Coate and Ravallion (1993), Ligon et al (2002), Genicot and Ray (2003)]. If this were the case in reality, one should expect full risk sharing in villages : this prediction is not confirmed empirically [6] [Deaton (1992) with data from Ghana and Thailand, Grimard (1997) with data from Côte d'Ivoire, Fafchamps and Lund (2003) with data from Philippines and so on]. Instead, there is evidence of partial risk sharing at the village level. This finding suggests that the village may not be the proper behavioral unit to study informal insurance. When Townsend (1994) analyzed data from three Indian villages he observed that informal insurance groups have limited sizes within the villages. He concluded by stating that the best way to model informal insurance is to follow an approach based on social networks. [7] Following Townsend (1994), authors have modeled informal insurance organizations as collections of bilateral agreements in village communities [De Weerdt and Dercon (2005), Fafchamps and Gubert (2007), Bramoullé and Kranton (2007), BGR, Ambrus et al. (2010)].

In the village, informal insurance does not systematically take place within limited

5. De Weerdt and Dercon (2005) find empirically evidence for multilateral norms in Tanzanian villages.

6. The only exception at our knowledge is the paper by Townsend (1994) who shows the existence of almost full risk pooling for consumption goods with data from Indian villages.

7. Some authors argue that the limited size of insurance groups is due to the cost of links (Murgai et al., 2002).

groups. Only individuals who trust each other in these groups so as to secure future reciprocity can do so. Based on informal insurance agreements in villages in Côte d'Ivoire, Fafchamps and Lund (2003) have highlighted two characteristics of these homogenous groups : they are governed by trust (trust is the first requirement to participate in an insurance group) and they are not disjoint but use to be overlapping. [8] In this chapter, we aim to build a model of informal insurance that accounts for most empirical findings. The objective is to be as realistic as possible. Therefore, we model informal insurance as multilateral arrangements within homogenous groups that overlap. Thus, a population (village) is modeled as an informal insurance organization. This is an institution composed of a collections of homogenous and overlapping groups, governed by multilateral norms. We proceed by modeling these homogenous groups (where informal insurance take place) as originating from trust relationship networks (that we consider exogenous to the model) that exist in the village. As these networks may overlap, individuals who belong to distinct networks, *overlapping individuals*, act as bridges to spread wealth across networks. These overlapping individuals help to share risk across homogenous groups. Notice that our multilateral approach embodies the traditional approaches using "clubs" or bilateral links. If the pre-existing trust relationship network is a complete network (all individuals are directly linked), our model replicates the "clubs" approach. On the other hand, if the pre-existing trust network is a tree (not more than two individuals are completely directly linked) our model replicates the bilateral links approach. The multilateral approach to model informal insurance is, to the best of our knowledge, the first attempt to nest the two traditional approaches and to accommodate overlapping insurance groups.

Within each homogenous group, sharing rules are defined to govern transfers. Since insurance is informal (no signed paper, no formal collateral, no legal court), some individuals may refuse to make transfers back. To survive (to be stable), an informal insurance arrangement needs to be "self-enforcing." That is, *the punishment*

8. Also, De Weerdt and Dercon (2005) observe on data from Tanzanian villages that insurance networks of villagers mostly overlap

that any person suffers for reneging on her prescribed duty discourages her from doing so. Some authors view this punishment as severe as to isolate the deviator and leave her in autarky. Others propose the *severance of only the link between deviators and their direct victims.*

In this chapter, we investigate "stability" of informal insurance organizations under each of these two punishment schemes and we derive results also for intermediate punishment schemes. This chapter can be viewed as a contribution to the literature on self-enforcing insurance arrangement, pointed out since 1980 by Postner and established at first by Kimball (1988). Using equal sharing in homogenous groups, we characterize stable informal insurance organizations without imposing extreme discounting. [9] In the particular case where individuals discount future at a very high or a very low rate, our model replicates existing results. For intermediate discounting, we provide a set interval characterization with a lower and upper bound. We show that all stable informal insurance organization lays between these bounds. We identify some network characteristics that correspond to a specific discounting. Defining preferences not only over consumption but also over social privileges, we propose a quantifier of individual *external cost* (the cost of losing social privileges). We find that high external costs reinforce stability. Our simulations identify parameters that reinforce stability such as : high level of output, high output variation, high risk aversion, high external costs. Finally we develop a procedure that equates consumption for linked individuals following idiosyncratic income shocks.

The chapter is organized as follows. In Section 4.2 we set the basics of the model and we derive stability results in Section 4.3. We derive static comparative results for exogenous parameters of the model and we provide numerical simulations in Section 4.4. In Section 4.5 we develop a procedure that equate consumption for linked individuals and finally in Section 4.6 we discuss some extensions and we conclude.

9. The bulk of the literature derive stability results for informal insurance by imposing high discounting

4.2 The model

In this section we build the model of multilateral risk sharing and we set conditions under which informal insurance arrangements are self-enforcing.

4.2.1 Endowment

Consider a population N of n individuals. At any date, a state of nature $\theta \in \Theta$ is realized with a probability $p(\theta)$. We assume that Θ is finite. In each state θ, an income distribution $y(\theta) = \left(y^i(\theta)\right)_{i \in N}$ is obtained. For each $i \in N$, $y^i(\theta) \in Y$, where $Y \subseteq \Re_+$ is a finite set. We assume that realized incomes are independent. [10]

4.2.2 Risk sharing groups based on networks

In our model, not all possible groups that can form, can also share risk. Only groups that originate from an existing trust relationship networks can do so. In fact, given a village, not all individuals can be engaged in reciprocal agreements with each other. Only individuals that trust each other in limited groups can do so. As highlighted by Fafchamps and Lund (2003) : "...during the interviews, many respondents stressed the role of trust building before gifts and loans can take place". To account for this empirical fact, we postulate a pre-existing trust relationship in the population. It is exogenous [11] to our model but it induces a network g. Risk sharing groups form given only this network g. In the sequel we theorize the formation of homogenous groups given a network.

Let N be the population (village). A *coalition* S is a non-empty subset of N. Two distinct coalitions S, S' are *overlapping* if $S \cap S' \neq \emptyset$. An *overlapping individual* is one that belongs to at least two overlapping coalitions. A *cover* γ of N is a collection

10. All our results hold under the weaker condition that realized incomes are not perfectly correlated.

11. We take the trust relationship network as exogenous to the model. We follow Fafchamps and Gubert (2006, with Philippines data) who find no evidence for strategic formation of risk sharing networks.

of coalitions, $S_1, S_2, \ldots, S_m$, such that $\bigcup_{k=1}^{m} S_k = N$.

A *network* g is a list of unordered pairs $(i,j) \in N \times N$, ij for simplicity, $i \neq j$. Let G be the set of all the networks. For all $g \in G$, define :

- $N(g) \equiv \{i \in N \mid \ \exists j \in N, ij \in g\}$
- $D(g) \equiv \{S \subseteq N(g) \mid \ \forall i,j \in S, ij \in g\}$ as the set of all fully directly connected subsets of $N(g)$
- $\mathcal{C}l(g) \equiv \{S \in D(g) \mid \ \nexists S' \in D(g) \ s.t. \ S \subset S'\}$ as the set of cliques : maximal directly connected subsets of $N(g)$
- $I(g) \equiv N \setminus N(g)$ as the set of all singletons that have no link in g
- a component C as a set of completely linked (directly or indirectly) elements of $N(g)$.

Definition 21. *Given $g \in G$, the cover representation of g denotes the set $\gamma_g \equiv I(g) \cup \mathcal{C}l(g)$.*

Remark 17. The definition of γ_g is indeed a representation of g. In Agbaglah (2011), we show that for all $g, g' \in G$, $g \neq g' \Rightarrow \gamma_g \neq \gamma_{g'}$. Thus, for all $g \in G$, we can use γ_g as the cover representation of g. Since γ_g is a collection of coalitions formed by individuals who are directly linked, we use it to represent the collection of homogenous overlapping groups originating from the network g. These groups are risk sharing groups within which insurance is organized. Notice that not all covers can be represented by networks. For example covers such that some coalitions are included in others cannot originate from networks.

In this chapter we consider only covers induced by networks. For example consider a village with two ethnic groups G_1 and G_2 without common individuals ($G_1 \cap G_2 = \emptyset$). Suppose that all individuals in G_1 are directly linked and so is the case in G_2. Furthermore, for some reason (marriage for example) one individual $i \in G_1$ becomes directly linked to all individuals in G_2. Two individuals who trust each other are directly linked. Let g denote the trust relationship network that obtains. From the cover representation, $\gamma_g = \{G_1, G_2'\}$ with $G_2' = \{i\} \cup G_2$, there are two informal insurance groups : G_1 and G_2'. Individual i is an overlapping individual.

We follow the empirical findings by Fafchamps and Lund (2003) that informal insurance homogenous groups are "institutions". They are governed by norms. We consider here multilateral norms unlike bilateral norms in BGR. In our setting, *norms* consist of *sharing rules* and *punishments.*

4.2.3 Multilateral sharing rules

A multilateral sharing rule is an extension of a bilateral sharing rule (bilateral norm in the setting of BGR) to more than two persons. In our setting, given $g \in G$, a *multilateral sharing rule* specifies individual consumption within coalitions of γ_g which are risk sharing groups originating from g. Each individual $i \in N$ can only make transfers to (or receive transfers from) a coalitional member. Notice that members of the same coalition are all directly linked. Only overlapping individuals can make (or receive) transfers across coalitions. Direct links from a pre-existing trust relationship network convey transfers. As a result, there is no transfer flows across components (a component is the set of completely linked –directly or indirectly linked– individuals). For simplicity, we assume in the remainder that the same multilateral sharing rule applies in coalitions belonging to the same component. We represent received transfers by positive numbers and sent transfers by negative numbers.

Let $S \in \gamma_g$ be a coalition. Let $\theta \in \Theta$ be a state of nature. Let $z^i_S(\theta)$ denote the net transfer from i within the coalition S. The overall transfer from an individual i is denoted $z^i(\theta)$ and defined by $z^i(\theta) = \sum_{i \in S,\ S \in \gamma_g} z^i_S(\theta)$. Notice that if i is not an overlapping individual, $z^i(\theta) = z^i_S(\theta)$, $S \ni i$.

We assume that *within a coalition, each individual can observe not only the realized income but also the aggregate of transfers of all her coalitional members.* [12]

Let $c^i(\theta)$ denote i's consumption. Let $c_S(\theta) \equiv \left(c^i(\theta)\right)_{i \in S}$ denote a vector of individual

12. In the village, ethnic groups or clans are so limited that it is easy to observe each one's net wealth. The same assumption is made in the bilateral setting by BGR.

consumptions, $y_S(\theta) \equiv \left(y^i(\theta)\right)_{i\in S}$ a vector of realized incomes, and $z_S(\theta) \equiv \left(z^i(\theta)\right)_{i\in S}$ a vector of overall transfers for all individuals in a coalition S. If there is no danger of confusion, we will use variables without their argument θ.

Given a multilateral sharing in a coalition $S \in \gamma_g$, the distribution of individual consumptions in S is defined by :

$$c_S \equiv \left(c^i\right)_{i\in S} = \left(y^i + z^i\right)_{i\in S} .$$

According to the assumption below, each individual $i \in S$ observes the aggregate of transfers of all her coalitional members. We need to spend a little more time on the special case of overlapping individuals. Overlapping individuals can make (or receive) transfers outside of S. Therefore multilateral sharing rules need to be consistent in the distribution of individual consumptions (see BGR for consistency conditions on norms). In this chapter we are interested by two specific multilateral sharing rules that we define below.

Definition 22. *Let g be a network on a population N, $S \in \gamma_g$ a coalition, and y a realized income distribution.*

Equal sharing rule. *A multilateral sharing rule on S is the equal sharing rule if and only if for all $i \in S$, $c^i = \frac{1}{|S|}\sum_{l\in S} y^l$.*

Equal consumption rule. *A multilateral sharing rule on S is an equal consumption rule if and only if there exists a level of consumption $\overline{c}_S$ such that for all $i \in S$, $c^i = \overline{c}_S$.*

Remark 18.

– (i) According to the equal sharing rule all individuals in S equally share their aggregated income. Therefore, each individual in S consumes the average of the realized incomes within S.

– (ii) According to the equal consumption rule, each individual in S consumes the same amount of consumption good $\overline{c}_S$.

- (iii) Given a coalition S, an equal consumption rule is not necessarily an equal sharing rule over this coalition. Since overlapping individuals can make transfers outside of S, $\overline{c}_S$ can be different from the average of the realized incomes in S. Therefore equal sharing rule is unique but an equal consumption rule is defined by $\overline{c}_S$.
- (iv) The two multilateral norms in the definition verify consistency conditions as defined in BGR.

Lemma 11. *If S is a coalition such that none of its elements is overlapping, then a consistent multilateral sharing rule on S is an equal consumption rule if and only if it is the equal sharing rule.*

Proof.

The necessary condition is straightforward because equal sharing rule is a specific case of equal consumption rule where $\overline{c}_S = \frac{1}{s}\sum_{i \in S} y^i$, with $s = |S|$.
Now consider an equal consumption rule over S and the induced consumption $\overline{c}_S$. As there is no overlapping individual in S, $\sum_{i \in S} z^i = 0$ (there is no transfer out of the coalition). But, for each $i \in S$, we have $\overline{c}_S = y^i + z^i$. Therefore by summing up on all i in S, $s\overline{c}_S = \sum_{i \in S} y^i$. Hence, $\overline{c}_S = \frac{1}{s}\sum_{i \in S} y^i$. That shows the sufficient condition. □

Empirical works show that there is no full risk pooling at village level but only some reduced size groups can fully share risk [Townsend (1994), Ligon et al. (2002), Fafchamps and Lund (2003)]. Also, a desired sharing rule that minimizes the risk over income variation in a group, equates consumption for all individuals in that group. Thus, since no transfer is possible across components, *we can consistently investigate risk sharing only at the level of components*. According to this observation, how can we define a multilateral sharing rule on coalitions such that it equates consumption for all individuals in components ?

Proposition 10. *A consistent equal consumption rule within coalitions is equivalent to equal sharing rule for components.*

Remark 19. A similar result is obtained by BGR in the bilateral setting and is well expressed by De Weerdt and Dercon (2005) : "...it can be shown that if every household belongs to a network and all these networks overlap with each other (have some common members), then full insurance within the confines of the separate networks necessarily implies Pareto-efficiency at village-level".

Proof.

Equal sharing for all linked individuals means that each individual receives the same amount of consumption good that is the mean of the realized income. Therefore in every coalition, each individual consumes the same amount of consumption good. Thus, the sharing rule is an equal consumption rule.

On the other hand, a component can be viewed (in terms of transfer) as big coalition where the net transfer among no linked individuals is zero. Then according to the previous lemma, we obtain equal sharing for all the linked individuals. □

In the sequel, we consider equal sharing on components. The same assumption is used in Genicot and Ray (2002). This is a strong assumption. However, in Section 4.5, we develop a realistic procedure to obtain equal sharing at component level.

4.2.4 Preferences

Most of empirical papers find difficulties to explain the distribution of individual consumption following a shock in income, even if there is evidence for risk sharing. For example, Ligon et al. (2002) show that wealthier households tend to consume less than what they should. One explanation of that is the functional form of the utility function that is used. Unlike a bulk of the literature, we state that individuals may not only prefer consumption goods, but also have preferences over other goods that are not necessarily convertible into consumption. The example we have in mind is social privileges. In rural communities, people enjoy social privileges like : respect, visits, helps in one's farm, high influence on decision making, high audience in village assembly. It may be the case that wealthier individuals, even if they consume less by sharing goods with others, gain in return social privileges that they value. Therefore, their overall outcome is higher. Thus using preferences defined only over consumption may underestimate outcomes. We will generally denote these social privileges (or everything similar) as *externalities.*[13]

We endow each individual with a utility function that depends not only on the consumption good, but also on the architecture of the network. For example, the more the direct links, the more the social influence in the community (BGR address this as monotonicity with the idea that this is convertible into consumption good. Here we state that individuals value these social privileges but not as consumption). That pattern exhibits positive externalities. Following Ambrus et al. (2010), we state that these externalities arise not at the village level, but at the level of homogenous "islands" which are represented by coalitions in our setting. We borrow from Ambrus et al. (2010) that external effects are substitutes to the consumption good. The difference in our model resides in the fact that we do not allow here the external good to be convertible in consumption. Thus, we endow each individual with a separable additive function $U_i(c, g) = u_i(c) + f_i(g)$ where u_i is a smooth function,[14] strictly concave and increasing. For simplicity, in the sequel $u_i \equiv u$ for all i. The second part f_i is a smooth function, increasing in the number of directs links of i. For simplicity, $f_i(g) \equiv f(d_i(g))$, where $d_i(g)$ is the number of direct links of i. Furthermore, we set that $f_i(\emptyset) = f(0) = 0$: *an isolated individual (singleton) is not subject to external effects.*

Introducing externalities is not only motivated by descriptive perspectives. Later in the chapter, we show that externalities stand as compensation to overlapping individuals for their effort to spread wealth towards coalitions.

13. We use the word externalities in our model because we have in mind that trust relationships are not built strategically to share risk (also empirically found by Fafchamps and Gubert, 2006, with Philippines villages). Therefore, the side effects of social privileges (for example) come as externalities because insurance addresses initially consumption only.

14. We have in mind f defined over the continuous set of real numbers

Remark 20. Since transfers and externalities across components are no allowed in our model, in the remainder we limit ourselves to networks such that all individuals are directly or indirectly linked. [15] These are networks with only one component.

4.2.5 Enforcement constraints

Insurance is said to be informal because it is *not formally regulated* : no signed paper, no effective collateral, no court. Thus, in some realized states, some individuals who benefit from positive transfers in the past may find it suboptimal to make transfers back when they are called for that. Therefore they may renege on their prescribed duty. If that happens, we say that these individuals *deviate*. For an informal insurance arrangement to survive, we need some self-enforcement constraints in order to discourage such an opportunist behavior. The literature is not unanimous on the appropriate reaction following a deviation. The most common punishment schemes that exist in the literature are what BGR denote as the *strong punishment* and the *weak punishment*. [16]

Definition 23. *If an individual i deviates :*

- *The strong punishment consists of the severance of all links with i in all coalitions that victims belong to.*
- *The weak punishment consists of the severance of only the links between the deviator and the victims.*

Notice that in the sense of link severance, any other possible punishment stands between the strong punishment (the strongest possible) and the weak punishment (the weakest possible).

Formally, for all $i \in N$, let $\mathcal{S}^i \equiv (\bigcup_{S \ni i, S \in \gamma_g} S) \setminus \{i\}$. An individual i can only deviate

15. This limitation is made for simplicity. For networks that admit more than one components, risk sharing will be studied component by component. Remember that there is neither transfers, nor externalities across components. After all this assumption is not unrealistic. In Tanzanian villages for example, De Weerdt and Dercon (2005) noticed that there is no isolated subnetwork of households.

16. BGR explore also intermediate punishment schemes that we will discuss at the end of the chapter.

to a set $D \subseteq \mathcal{S}^i$. Let c^i_D denote i's current consumption if she deviates to a set D. Depending on the punishment, severance of links will induce a residual network g_R, and a residual consumption c^i_R. Thus, i decides to deviate if the gain from the pair (c^i, g) is less than the gain from the pair (c^i_D, g_R). Let $V^i(g)$ be the expected per period payoff of i from the next period on if she abides by the sharing rule, and $V^i(g_R)$ the expected per period payoff of i from the next period on if she deviates.
To discourage individuals to systematically deviate, the gain of deviating today should be less than the actual value of the loss following the punishment from the next day on.

Definition 24. *Given $g \in G$, an informal insurance organization (IIO_g in the remainder) is the institution defined by γ_g and a multilateral norm (multilateral sharing rule and punishment scheme) that governs behavior in the coalitions of γ_g.*

- *IIO_g is stable if and only if it is immune to individual deviations. Formally, IIO_g is stable if and only if*

$$(1-\delta)U_i\big(c^i(\theta_0), g\big) + \delta V^i(g) > (1-\delta)U_i\big(c^i_D(\theta_0), g\big) + \delta V^i(g_R)$$

  for all $\theta_0 \in \Theta$, $i \in N$, $D \subseteq \mathcal{S}^i$, and $g_R \subset g$.
- *IIO_g is weak-stable if it is stable and the punishment scheme is the strong punishment.*
- *IIO_g is strong-stable if it is stable and the punishment scheme is the weak punishment.*

In the remainder, we denote by β the tuple $\beta \equiv (\theta_0, i, D, g_R)$, and $\mathcal{B}$ the collection of all such tuples.

4.3 Stability

Stability of an informal insurance organization, in the sense of no deviation, depends on the punishment scheme. In this section we investigate stability under weak punishment and also under strong punishment. Remember that any other punishment scheme is in between these two. Specifically we provide conditions for either stability and we investigate how this stability is affected by some factors : discounting, network architecture, preferences.

Lemma 12. *For all $g \in G$, IIO_g is stable if and only if*

$$U_i\big(c^i_D(\theta), g\big) - U_i\big(c^i(\theta), g\big) < \frac{\delta}{(1-\delta)}[V^i(g) - V^i(g_R)]$$

The same result is obtained in different contexts by Coate and Ravallion (1993), and by BGR. The clue is that, an IIO is stable if and only if following the prescribed sharing rule is more worthy than cheating. This is nothing but the conditions for a self enforcing stable agreement.

4.3.1 Characteristics of stable IIOs

In this section we provide general conditions for stability. When contemplating deviation, an individual takes into account not only the loss of future coverage due to punishments that will follow her action, but also losses in terms of external effects (social privileges).

Proposition 11. *IIO_g is stable if and only if for all $\beta \in \mathcal{B}$,*

$$u\big(c^i_D(\theta_0)\big) - u\big(c^i(\theta_0)\big) < \frac{\delta}{1-\delta}\Big\{\sum_{\theta} p(\theta)\big[u\big(c^i(\theta)\big) - u\big(c^i_R(\theta)\big)\big]\Big\} + \frac{\delta}{1-\delta}\big\{f\big(d_i(g)\big) - f\big(d_i(g_R)\big)\big\} \tag{4.1}$$

Interpretation.

This proposition shows indeed the existence of two consequences following deviation. The first *direct* consequence is the loss of consumption smoothing coverage. The second *indirect* consequence is the loss of social privileges as external effects. For example an individual who values more social privileges will be less tempted to deviate, even if in terms of consumption, deviation will not harm him as much as making his prescribed transfer. After all, Proposition 12 shows that IIO_g is stable if and only if the gain from defection today in term of consumption is less than the actual value of the combined effect of both direct and indirect consequences.

Proof.

IIO_g is stable $\Leftrightarrow \forall \beta \in \mathcal{B}$,

$U_i\big(c_D^i(\theta), g\big) - U_i\big(c^i(\theta), g\big) < \frac{\delta}{(1-\delta)}[V^i(g) - V^i(g_R)]$

$\Leftrightarrow u\big(c_D^i(\theta_0)\big) - u\big(c^i(\theta_0)\big) < \frac{\delta}{1-\delta}\Big\{ \sum_\theta p(\theta)\big[u\big(c^i(\theta)\big) - u\big(c_R^i(\theta)\big)\big] + f\big(d_i(g)\big) - f\big(d_i(g_R)\big)\Big\}$,

obtained from $U_i(c, g) = u(c) + f\big(d_i(g)\big)$. □

Let g be a network and γ_g the cover representation of g. We denote by $\mathcal{SS}$ the set of all networks g such that IIO_g is strong-stable and by $\mathcal{WS}$ the set of all networks g such that IIO_g is weak-stable.

Proposition 12. *For all $g \in G$, if γ_g is strong-stable, then γ_g is weak-stable. The converse is not true. This is $\mathcal{SS} \subseteq \mathcal{WS}$.*

Proof.

The proof is by contraposition. Suppose that IIO_g is not weak-stable. There exists $\beta = (\theta_0, i, D, g_R) \in \mathcal{B}$, such that

$$u\big(c_D^i(\theta_0)\big) - u\big(c^i(\theta_0)\big) \geq \frac{\delta}{1-\delta}\Big\{ \sum_\theta p(\theta)\big[u\big(c^i(\theta)\big) - u\big(c_R^i(\theta)\big)\big]\Big\} + \frac{\delta}{1-\delta}\{f_i(g) - f_i(g_R)\},$$

where g_R is the network obtained after the strong punishment following i's deviation to D. Let $g_{R'}$ denote the network that obtains in the same context with the weak punishment. Obviously, $g_R \subseteq g_{R'}$, and the expected residual consumption for i following the weak punishment, $\sum_\theta p(\theta)u\big(c_{R'}^i(\theta)\big)$, is obviously not less than what obtains in the strong punishment case. Therefore, $f\big(d_i(g_R)\big) \leq f\big(d_i(g_{R'})\big)$, and $\sum_\theta p(\theta)u\big(c_R^i(\theta)\big) \leq \sum_\theta p(\theta)u\big(c_{R'}^i(\theta)\big)$ since incomes realizations are independent.

Thus,

$$u\big(c^i_D(\theta_0)\big)-u\big(c^i(\theta_0)\big) \geq \frac{\delta}{1-\delta}\Big\{\sum_\theta p(\theta)\big[u\big(c^i(\theta)\big)-u\big(c^i_{R'}(\theta)\big)\big]\Big\}+\frac{\delta}{1-\delta}\{f(d_i(g))-f(d_i(g_{R'}))\}.$$

Hence, IIO_g is not strong-stable.

Counterexample

The counterexample is obtained for g as the complete network of 3 individuals, $f(x) = x$, two possible income realizations 0 and 1 with equal probability, $u(c) = c - \lambda c^2$, $\lambda = 0.4$, $\delta = 0.2$. □

It is interesting to know the conditions under which the converse is true. Even if it is no obvious to specify these condition, the following corollary identifies some networks where the equivalence holds. A *tree* is a network with no cycles. If g is a tree, the highest completely connected sets are of size 2.

Corollary 4. *If g is a tree, then IIO_g is strong-stable if and only if it is weak-stable.*

Proof.

The proof relies on the fact that if g is a tree, then strong and weak punishments result in the same residual network g_R. The conclusion comes from the previous proof. □

Let $\mathcal{PS}$ denote the set of networks g such that IIO_g is stable under any other intermediate punishment scheme $\mathcal{P}$. How can we compare $\mathcal{PS}$ to $\mathcal{SS}$ and $\mathcal{WS}$.

Corollary 5. $\mathcal{SS} \subseteq \mathcal{PS} \subseteq \mathcal{WS}$

Proof.

Following the proof of Proposition 13, for any punishment scheme P and the residual network $g_{R"}$, we have $g_R \subseteq g_{R''} \subseteq g_{R'}$. Thus the results follows from the same steps as in the proof of Proposition 13. □

From this result above, we obtain a set interval of networks that induce stable IIOs. The lower bound is $\mathcal{SS}$ and the upper bound is $\mathcal{WS}$.

Discounting is a key factor in the evaluation of deviation gain versus punishment loss when an individual contemplates deviation. As this is an intertemporal arbitrage, the discount factor matters for stability. Most of the economics literature on informal insurance studies stability only under extreme values of δ. Most of the time δ is assumed close to unity. This is very restrictive since experimental studies show that δ varies widely among individuals and geographical regions. To accommodate all possible economies, we investigate stability without imposing restrictions on δ.
First, we make the following observations.

Observation 1. Extreme values of discounting

Everything equal,

$\diamond$ If δ is high enough, then for all $g \in G$, each IIO_g is stable (Kimball, 1988, Ligon et al., 2002, BGR weak stability for all networks and strong stability for trees).

$\diamond$ If δ is low enough, then no IIO is stable. [17]

Proof.

Fix $\beta \in \mathcal{B}$, and let

$F_\beta(\delta) \equiv (1-\delta)\left[u\left(c^i_D(\theta_0)\right) - u\left(c^i(\theta_0)\right)\right] - \delta\left\{ \sum_\theta p(\theta)\left[u\left(c^i(\theta)\right) - u\left(c^i_R(\theta)\right)\right] + f\left(d_i(g)\right) - f\left(d_i(g_R)\right)\right\}$.

Notice that $u\left(c^i_D(\theta_0)\right) - u\left(c^i(\theta_0)\right) > 0$ because if i deviates, she consumes more than the prescribed amount of consumption. Also, $\sum_\theta p(\theta)\left[u\left(c^i(\theta)\right) - u\left(c^i_R(\theta)\right)\right] > 0$ because the residual expected utility over consumption is less than the expected utility corresponding to the whole set N. Finally, $f\left(d_i(g)\right) - f\left(d_i(g_R)\right) > 0$ because f is decreasing. From Proposition 13, IIO_g is stable iff for all $\beta \in \mathcal{B}$, $F_\beta(\delta) < 0$.

For a fixed β, F_β is differentiable and decreasing. Furthermore, use the continuity of F_β in the interval $[0,1]$ and the fact that $F_\beta(0) > 0$ and $F_\beta(1) < 0$ to obtain that

17. This observation is not common in economics literature on informal insurance because attention is rather focused on highest values of δ.

there exists δ_β and δ'_β respectively such that $F_\beta(\delta_\beta) > 0$ and $F_\beta(\delta'_\beta) < 0$. As $\mathcal{B}$ is finite, take respectively the lowest δ'_β for the second part of Observation 1, and the highest δ'_β for the first part of Observation 1. $\square$

As it is mostly the case in the literature, the two results in Observation 1 are obtained under the strong assumptions of $\delta \to 1$ or $\delta \to 0$. However, for intermediate values of δ stability is not obvious. This depends mostly on the characteristics of the preferences, and the characteristics of g. Since discounting is an individual behavior, one need to derive stability results without assumptions on the value of δ.

4.3.2 Non-extreme values of the discount factor

Given a punishment scheme, Proposition 12 suggests that stability depends on some other factors including discounting, the architecture of the network g, and preferences (utility over consumption, externalities).

Stability and discount factor

In Observation 1, we show that for lower values of δ, $\mathcal{SS} = \mathcal{WS} = \emptyset$ whereas for higher values of δ, $\mathcal{SS} = \mathcal{WS} = \mathcal{G}$. We can guess from this result a positive relationship between discounting and the size of stable IIOs. The following shows formally this monotonic result and also the existence of a threshold value of δ from where IIO_g is stable for all $g \in G$.

Corollary 6. *The number of networks that induce stable IIOs increases with δ. Furthermore, for all $g \in G$, there exists $\delta_0 \in (0,1)$ such that for all $\delta \in (\delta_0, 1)$, IIO_g is stable.*

Proof.

We prove first the second part of the corollary.

From the previous proof, for a fixed β, F_β is continuous and decreasing on $[0,1]$,

$F_\beta(0) > 0$, and $F_\beta(1) < 0$. By the intermediate value theorem there exists $\delta_\beta \in [0,1]$ such that $F_\beta(\delta_\beta) = 0$. As $\mathcal{B}$ is finite, let δ_0 be the maximum value of such δ_β's. F_β being decreasing, for all $\delta > \delta_0$, $F_\beta(\delta) < 0$ and this holds for all β. Therefore, IIO_g is stable for all $g \in G$.

The first part of the corollary is due to the fact that for a fixed β, F_β is decreasing. Let IIO_g^k be the IIO_g of a community which discounts the future at the rate δ^k, $k = 1, 2$ with $\delta^2 > \delta^1$. We have IIO_g^1 is stable iff for all $\beta \in \mathcal{B}$, $F_\beta(\delta^1) < 0$. As $\delta^2 > \delta^1$, $F_\beta(\delta^2) < 0$. Thus IIO_g^2 is stable. □

Notice that the threshold value δ_0 defined in Corollary 6 depends on the network g. It is interesting to identify networks that correspond to some specific values of this threshold value.

Theorem 10. *Stability is obtained with the lowest δ_0 if and only if g is the complete network.*

Proof.

From the proof of Observations 1, given a network g, $\delta_0 = Max_\beta\{\delta_\beta\}$.

First, let g be the complete network and $g' \neq g$. For all $i \in N$, $d_i(g) \geq d_i(g')$. As f is decreasing, for all $i \in N$, $f(d_i(g)) \geq f(d_i(g'))$. Since D defines c_R^i, and $d_i(g_R)$, everything equal, $\frac{u\left(c_D^i(\theta_0)\right)-u\left(c^i(\theta_0)\right)}{u\left(c_D^i(\theta_0)\right)-u\left(c^i(\theta_0)\right)+\sum_\theta p(\theta)\left[u\left(c^i(\theta)\right)-u\left(c_R^i(\theta)\right)\right]+f\left(d_i(g)\right)-f\left(d_i(g_R)\right)} \leq \frac{u\left(c_D^i(\theta_0)\right)-u\left(c^i(\theta_0)\right)}{u\left(c_D^i(\theta_0)\right)-u\left(c^i(\theta_0)\right)+\sum_\theta p(\theta)\left[u\left(c^i(\theta)\right)-u\left(c_R^i(\theta)\right)\right]+f\left(d_i(g')\right)-f\left(d_i(g_R)\right)}$. Therefore by maximizing over all $\beta \in \mathcal{B}$, we obtain a lower δ_0 with g. As g' is arbitrary, then δ_0 is the lowest possible.

Second, $\delta_\beta = \frac{u\left(c_D^i(\theta_0)\right)-u\left(c^i(\theta_0)\right)}{u\left(c_D^i(\theta_0)\right)-u\left(c^i(\theta_0)\right)+\sum_\theta p(\theta)\left[u\left(c^i(\theta)\right)-u\left(c_R^i(\theta)\right)\right]+f\left(d_i(g)\right)-f\left(d_i(g_R)\right)}$.

Let $A \equiv u\big(c_D^i(\theta_0)\big) - u\big(c^i(\theta_0)\big)$, $B \equiv \sum_\theta p(\theta)\big[u\big(c^i(\theta)\big) - u\big(c_R^i(\theta)\big)\big]$, and $C \equiv f\big(d_i(g)\big) - f\big(d_i(g')\big)$. Therefore the candidate g is such that A is minimal and $B + C$ is maximal because δ_β is increasing in A and decreasing in $B + C$. As D defines C_R^i, and $d_i(g_R)$, if we maximize over θ_0 and D, we obtain $c_D^i = y_h$, the highest income in Y, for θ_0

such that only i gets y_h and all others get y_l, the lowest income in Y. Thus $d_i(g_R) = 0$ and the residual set is the singleton set. Therefore, the lowest value of δ_0 is obtained for the network g such that for all i, $d_i(g)$ is maximal. This is $d_i(g) = n - 1$ and this is nothing but the complete network where each individual has exactly $n - 1$ direct links. □

In the theorem, we identify a relation between complete networks and δ_0. As similar result is found by BGR in the bilateral setting. It is interesting to generally investigate possible correlations between the architecture of the network g and δ_0. We start this investigation by the following observation.

Observation 2.

◇ For weak stability, the following characteristics of the preexisting network g influence δ_0 :

(i) the number of individual links in g is decreasing in δ_0 ;

(ii) the number of overlapping individuals in g is increasing in δ_0 ;

(iii) the number of overlapping individuals in g is decreasing in δ_0.

◇ For strong stability, apart from the previous characteristics of the preexisting network g that influence δ_0, the number of distinct overlapping individuals in the same clique in g is positively correlated to δ_0.

Proof.

From $F_\beta(\delta)$ defined in the proof of the observations, we obtain
$F(\delta^*) = 0 \Leftrightarrow \delta^* = \frac{u\left(c^i_D(\theta_0)\right) - u\left(c^i(\theta_0)\right)}{u\left(c^i_D(\theta_0)\right) - u\left(c^i(\theta_0)\right) + \sum_\theta p(\theta)\left[u\left(c^i(\theta)\right) - u\left(c^i_R(\theta)\right)\right] + f\left(d_i(g)\right) - f\left(d_i(g_R)\right)}$.
For a fixed β, the solution δ^* exists and is in $(0,1)$ because the denominator is positive and is greater than the positive numerator. Take $\delta_0 \equiv Max_\beta\{\delta^*\}$. As $\mathcal{B}$ is finite, δ_0 exists. This value δ_0 depends on g. Let $A \equiv u\left(c^i_D(\theta_0)\right) - u\left(c^i(\theta_0)\right)$, $B \equiv \sum_\theta p(\theta)\left[u\left(c^i(\theta)\right) - u\left(c^i_R(\theta)\right)\right]$, and $C \equiv f\left(d_i(g)\right) - f\left(d_i(g')\right)$. Therefore the

candidate g is such that A is minimal and $B+C$ is maximal because δ^* is increasing in A and decreasing in $B+C$.

• Weak stability.

If $i \in N$ is a non-overlapping individual, A, B and $d_i(g_R)$ are constant for all g. Therefore the candidate g is such that $d_i(g)$ is the highest for all non-overlapping individuals.

If $i \in N$ is an overlapping individual, the result is not straightforward. Networks that minimize A, maximize B if and only if there are several cliques and few other overlapping individuals.

• Strong stability.

Apart from the previous conditions for weak stability, add that networks that minimize A are with few links. If we fix $B+C$ to be maximal, then each individual deviates to all the linked parterres in case of deviation. Therefore, the networks with highest $d_i(g)$ for all is are the candidates. □

The combined effect of characteristics enumerated in Observation 2 is unknown. This combined effect depends on the parameters of the functionals u and f. To fix ideas on this combined effect, we simulate stability conditions for six distinct 8-person networks (Figure 4.1). [18] We select three network characterizations that embody the influential factors. These are *sparseness*, *clustering* and *density*. The sparseness coefficient of a network quantifies how sparse it is. [19] The density keeps track of the relative fraction of links that are present in the network. The clustering is a measure of cliquishness, that accounts for the number and size of cliques of a networks. [20]

As we can see from Tables 4.1 and 4.2, sparseness seems to be irrelevant for

18. The first five networks are obtained thank to BGR
19. See BGR for details on Sparseness.
20. Details of these measures can be found in Jackson (2008).

stability.

For weak stability, apart from the circle, number of links and density characterize tend to be negatively correlated to δ_0. The combined effect of these characteristics places the circle (8 overlapping individuals and 8 links) below the star (1 overlapping individual and 7 links) and the simple tree (7 overlapping individual and 7 links).

For strong stability, apart from the bridge (2 overlapping individuals and 13 links) and the circle, number of links, density, and clustering tend to be negatively correlated to δ_0.

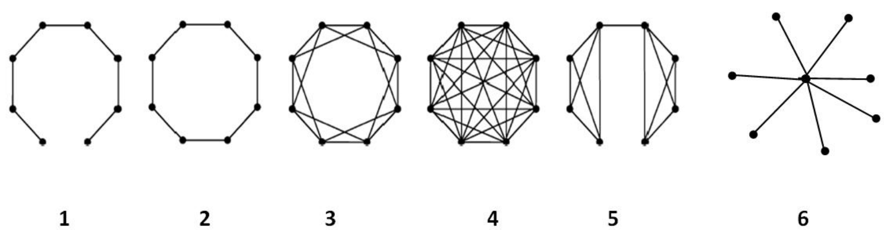

FIGURE 4.1 – Examples of 8-person networks : (1) Simple tree, (2) Circle, (3) Two-neighbors, (4) Complete, (5) Bridge, (6) Star

Network		**Characteristics**			δ_0
	Sparseness	**Links**	**Clustering**	**Density**	
Complete	1	28	1	1.00	0.18
Two-neighbors	2	16	0.5	0.57	0.28
Bridge	1	13	0.8	0.46	0.53
Star	0	7	0	0.25	0.56
Simple tree	0	7	0	0.25	0.58
Circle	6	8	0	0.29	0.62

TABLE 4.1 – Weak stability, network characteristics, and discount factor

Network	Characteristics				δ_0
	Sparseness	**Links**	**Clustering**	**Density**	
Complete	1	28	1	1.00	0.26
Two-neighbors	2	16	0.5	0.57	0.54
Star	0	7	0	0.25	0.56
Simple tree	0	7	0	0.25	0.58
Bridge	1	13	0.8	0.46	0.58
Circle	6	8	0	0.29	0.62

TABLE 4.2 – Strong stability, network characteristics, and discount factor

To conclude, the simulation results confirm what we formally show, that the complete network corresponds to the lowest δ_0. Furthermore, the reversion of the Bridge and the Star in the ordering of δ_0 from the weak-stability and the strong-stability suggests that there is no obvious monotonic relationship between network architecture and stability.

Network architecture, preference, and stability

In the previous section, we observe that the relationship between the architecture of the network and stability, depends on the punishment scheme. In the following theorem we provide comparative statics for weak stability.

Theorem 11. *Consider a network $g \in G$. Let g^1 be a network obtained from g by adding links such that the number of overlapping individuals is not greater than in g. If IIO_g is weak-stable, then IIO_{g^1} is also weak-stable.*

Proof.

First, let i be a non-overlapping individual. We write the stability condition as follows. Let θ_0 be a state where the incentive to deviate is the highest,

$-\frac{\delta}{1-\delta} f\big(d_i(g)\big) < u\big(c^i(\theta_0)\big) + \frac{\delta}{1-\delta} \sum_\theta p(\theta) u\big(c^i(\theta)\big) - \big[u\big(c^i_D(\theta_0)\big) + \frac{\delta}{1-\delta} \sum_\theta p(\theta) u\big(c^i_R(\theta)\big) + \frac{\delta}{1-\delta} f\big(d_i(g_R)\big)\big].$

In case of deviation, i lives in autarky from the next period on according to the strong punishment. For this reason, she deviates to all possible linked partners. Thus, the RHS of the inequation is constant. From the facts that f is increasing and that $d_i(g^1) \geq d_i(g)$, the stability condition obtains for all such i with g^1.

Second, let i be an overlapping individual. Individual i faces the same conditions as for g. Furthermore her incentive to deviate is less because f is increasing and $d_i(g^1) \geq d_i(g)$. □

Corollary 7. *If g is a circle and IIO_g is weak-stable, then each $IIO_{g'}$ such that g is a subnetwork of g' is also weak-stable. Especially the IIO obtained from the complete network is also weak-stable.*

Proof.

It is sufficient to notice that the number of overlapping individuals in g' is not greater than g. Furthermore, $g' = Max_{g''}\{g'' \supseteq g\}$ is the complete network. □

In the following we investigate how preferences over consumption affect stability. To begin, we make the following observations.

Observation 3 : preferences over consumption

⋄ Individuals in weak-stable IIOs are characterized by individuals with high aversion to risk and low marginal utility over consumption.

⋄ Apart from the previous characteristics, individuals in strong-stable IIOs are characterized by high valuations of social privileges (external effects).

Remark 21. Observation 3 is a natural result. In fact, highly risk averse individuals value more smooth consumption. Therefore, their incentive to deviate is low as the punishment becomes more and more severe. On the other hand, if the punishment is weak, according to equal consumption, an individual can sever enough links without damaging consumption smoothing coverage. Therefore stability is reinforced by the external effects only.

Proof.

• Weak stability : strong punishment

If i is a non-overlapping individual, $\frac{\delta}{1-\delta}\Big\{\sum_\theta p(\theta)\big[u\big(c^i(\theta)\big) - u\big(c^i_R(\theta)\big)\big] + f\big(d_i(g)\big) - f\big(d_i(g_R)\big)\Big\} \equiv Constant$. Therefore the stability condition is equivalent to $u\big(y^i(\theta_0)\big) - u\big(\overline{c}(\theta_0)\big) < Constant$. Because according to the strong punishment, in case of deviation, i deviates to all. The second order Taylor expansion yields $u(y^i) - u(\overline{c}) \simeq (y^i - \overline{c})u'(\overline{c}) + \frac{1}{2}(y^i - \overline{c})^2 u''(\overline{c})$. This implies that $u(y^i) - u(\overline{c}) \simeq (y^i - \overline{c})u'(\overline{c})\big[1 - \frac{1}{2}(y^i - \overline{c})\frac{u''(\overline{c})}{u'(\overline{c})}\big]$. The quotient $\frac{u''(\overline{c})}{u'(\overline{c})}$ is the Arrow-Pratt absolute risk aversion coefficient.

If i is a overlapping individual, the stability condition is equivalent to $\big[u\big(c^i_D(\theta_0)\big) - u\big(\overline{c}(\theta_0)\big)\big] + \frac{\delta}{1-\delta}\big[\sum_\theta p(\theta)u\big(c^i_R(\theta)\big) + f\big(d_i(g_R)\big)\big] < Constant$. For the same level of c^i_R and $d_i(g_R)$, the LHS is low if $u\big(c^i_D(\theta_0)\big) - u\big(\overline{c}(\theta_0)\big)$ is low.

• Strong stability : weak punishment

In this case, $\big[u\big(c^i_D(\theta_0)\big) - u\big(\overline{c}(\theta_0)\big)\big] + \frac{\delta}{1-\delta}\big[\sum_\theta p(\theta)u\big(c^i_R(\theta)\big) + f\big(d_i(g_R)\big) - f\big(d_i(g)\big)\big] < Constant$. Apart from what we describe in the weak stability case, the key element for stability here is $-f(d_i(g)$. For a given level of $d_i(g)$, high $f(\cdot)$ yields low LHS and therefore, strong stability. □

In our setting, individual preferences are defined not only on consumption but also on social privileges. Therefore, we investigate in the next section the impact of externalities on stability.

Stability and externalities

Consider a benchmark model with no externalities. Given $g \in G$, IIO_g is stable if and only if for all $\theta_0 \in \Theta$, $i \in N$, and $D \subseteq \mathcal{S}^i$,

$$u\big(c^i_D(\theta_0)\big) - u\big(c^i(\theta_0)\big) < \frac{\delta}{1-\delta}\Big\{\sum_\theta p(\theta)\big[u\big(c^i(\theta)\big) - u\big(c^i_R(\theta)\big)\big]\Big\}, \qquad (4.2)$$

First, if we consider weak punishment, the incentive to deviate is very high in a model without externalities. As far as a the residual network following a deviation indirectly connect an individual to others, she will deviate whenever she has a high realized income : the punishment will not harm her in terms of future consumption. The reason is that it is sufficient to have only one link to be indirectly connected to the whole community and benefit from smooth consumption coverage. Thus, the only possible stable IIOs are such that the preexisting network is a tree. This result is also obtained by BGR and Bramoullé and Kranton (2007). But in real life, not only tree-based IIOs are stable ! In a model with externalities, if an individual breaks a link, even if she has the same average consumption in the future, she will suffer from the lack of social privileges (external effects).

Secondly, if we consider strong punishment, each individual who belongs to only one coalition will hardly deviate. In fact when contemplating deviation, she finds that not only will she lose consumption smoothing, but also social privileges (external effects). Furthermore, overlapping individuals are key actors of risk pooling. For example in a model without externalities, an overlapping individual who is the only one to convey transfers from one coalition to another has a high incentive to deviate. He can collect transfers from one collection and keep them instead of giving them back to another coalition. If overlapping individuals keep playing their role of wealth spreading across coalitions, it may be because they find compensations for that. One way to deal with that is to impose exogenously that overlapping individuals must not deviate. But, since we want a model that is completely self-enforcing, imposing externality solves the case. With externalities, an individual who belongs to several coalitions has several direct links and somehow she values these links. If the external effect is high enough, no overlapping individual has the incentive to deviate.

We quantify externalities by the utility loss following a link severance. The idea is that if an individual values direct links, she suffers from losing them. In practice, whenever an individual lose a link, she lose part of her social privileges. Formally, let

$\mathcal{C} \equiv Max_{x\in\{1,2,\ldots,n\}}\{f(x)-f(x-1)\}$ be the maximal utility cost of losing one link for the same amount of consumption. We denote $\mathcal{C}$ by *external cost*. Since f is increasing, C is positive. Notice that if f is concave, $\mathcal{C} = f(1)$ because $f(0) = 0$.

To view the importance of the external cost, Figure 3 below shows an illustration obtained from an IIO $\{S_1, S_2\}$, with $S_1 \cap S_2 = \{i\}$. It turns out that in a model without externalities, the overlapping individual i has a high incentive to deviate in the coalition with the lowest per-capita income realization. By introducing externalities and increasing the external cost, there is a level of externalities where everything equal, no individual can afford consequences of a deviation. From this threshold on, the IIO is stable. Parameters of the illustration are : $Y\{0,1\}$, $u(c) = c - \lambda c^2$, $\lambda = 0.49$, $\delta = 0.8$,$|S_1| = 10$, $|S_2| = 2$. There are two levels of income, h=1 with probability p=0.5 and l=0. The external part is obtained by $f(d_i) = \alpha \times d_i$. Thus, $\mathcal{C} = \alpha$.

As we can see on Figure 4.2, there is no stability for small values of $\mathcal{C}$ until $\mathcal{C}$ reaches a threshold. The Figure 4.2 suggests a positive relationship between external cost and stability.

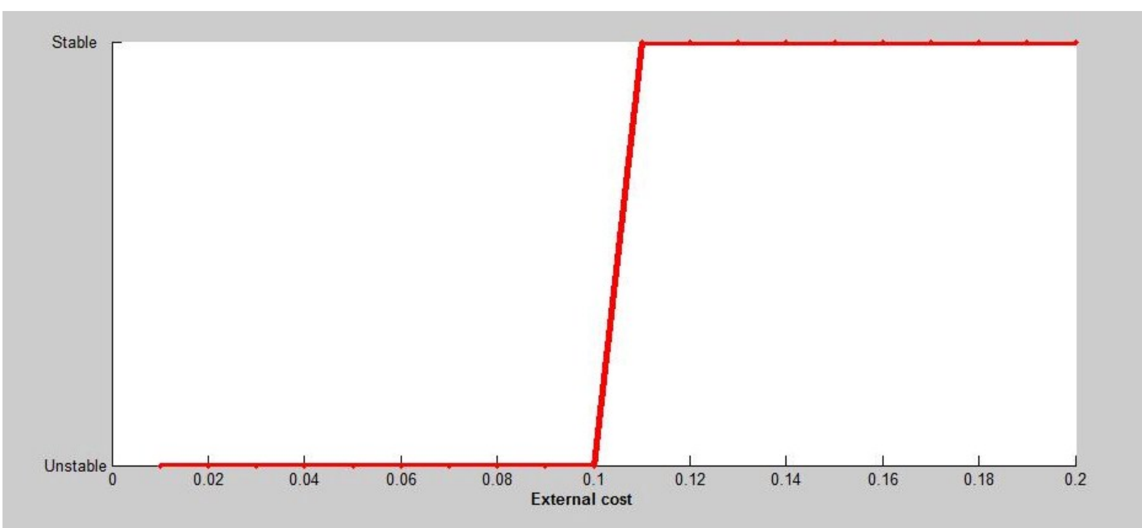

FIGURE 4.2 – Stability and external effects

In the following we show that the positive relationship between the external cost and stability that we observe on Figure 4.2 is a general result.

Theorem 12. *If the external cost $\mathcal{C}$ is high enough, then all IIO_g is stable for all $g \in G$.*

Proof.
The stability condition
$u\big(c^i_D(\theta_0)\big) - u\big(c^i(\theta_0)\big) < \frac{\delta}{1-\delta}\Big\{ \sum_\theta p(\theta)\big[u\big(c^i(\theta)\big) - u\big(c^i_R(\theta)\big)\big] + f\big(d_i(g)\big) - f\big(d_i(g_R)\big)\Big\}$
implies that
$u\big(c^i_D(\theta_0)\big) - u\big(c^i(\theta_0)\big) + \frac{\delta}{1-\delta}\sum_\theta p(\theta)u\big(c^i_R(\theta)\big) < \frac{\delta}{1-\delta}\big\{ \sum_\theta p(\theta)u\big(c^i(\theta)\big) + f\big(d_i(g)\big) - f\big(d_i(g_R)\big)\big\}$.
The LHS is a functional of the realized state of the nature. Since the number of states is finite, let $M \equiv Max_{\theta_0}\Big\{u\big(c^i_D(\theta_0)\big) - u\big(c^i(\theta_0)\big) + \frac{\delta}{1-\delta}\sum_\theta p(\theta)u\big(c^i_R(\theta)\big)\Big\}$. Also notice that $L = \sum_\theta p(\theta)u\big(c^i(\theta)\big)$ is constant. Thus, the stability condition becomes $f\big(d_i(g)\big) - f\big(d_i(g_R)\big) > K$ where $K = \frac{\delta}{1-\delta}(M - L)$. If there is a deviation by i, $d_i(g_R) \leq d_i(g) - 1$. Thus, $f\big(d_i(g)\big) - f\big(d_i(g_R)\big) \geq f\big(d_i(g)\big) - f\big(d_i(g) - 1\big)$. The RHS of this inequality is constant if f is a linear functional. Otherwise, let the external effect be determined by the maximum value of the RHS computed over $1, \ldots, n$. This is $\mathcal{C}$. It is sufficient to have $\mathcal{C} > K$ to obtain stability. □

The theorem above shows that introducing externalities in the model is an important feature and can explain the puzzling effect obtained by Ligon et al. (2002) that wealthier households tend to consume less. As far as wealthier households also have higher external cost, the puzzle is solved.

4.4 Application

In this section we investigate the role of exogenous parameters of the model regarding stability. Specifically, we study the impacts on stability of : income level, probability of income realizations, risk aversion, and external cost. In order to isolate these impacts, we consider a specific preexisting network. The architecture of this network is designed to summarize as simply as possible all the key features of multiple overlapping networks.

We borrow from De Weerdt and Dercon (2005) a specification of the preexisting network g that is composed of two complete subnetworks connected by one overlapping individual. The corresponding cover representation is $\gamma_g \equiv \{N_1, N_2\}$ with $|N_j| = n_j$ and $N_1 \cap N_2 = \{o\}$. Individual o is the only overlapping individual. We consider two levels of income, y_h realized with probability p, and y_l with probability $1-p$ such that $y_h > y_l > 0$. The utility function is by Bramoullé and Kranton (2007), $u(c) = c - \lambda c^2$. The coefficient λ increases with the risk aversion. For u to be strictly increasing and concave, we need $0 < \lambda < \frac{1}{2y_h}$. We consider a linear function to model externalities : $f(x) = \alpha x$, with $\alpha > 0$. Thus the external cost is $\mathcal{C} = \alpha$. For simplicity, we set $y_l \equiv y$ and $y_h \equiv y + d$, with $d > 0$.

The set of parameters we examine in the sequel is $\mathcal{P} \equiv \{y, d, p, \lambda, \alpha, n_1, n_2\}$.

4.4.1 Comparative static properties

The following lemma establishes the stability condition for the application.

Lemma 13.
Given $g \in G$, IIO_g is stable if and only if for all $k \in \{n, n_1, n_2\}$:
For weak-stability,

$$\textbf{\textit{(i)}} \quad d\left(\frac{k}{n} - \frac{1}{n}\right)\Big[1 - \lambda\big(2y + d\left(\frac{k}{n} + \frac{1}{n}\right)\big)\Big] < \frac{\delta}{1-\delta}\Big\{\lambda\left(\frac{1}{n-k+1} - \frac{1}{n}\right)d^2 p(1-p) + \alpha(n_j - 1)\Big\}.$$

For strong-stability, (i) and the additional condition,

$$\textbf{\textit{(ii)}} \quad d\left(\frac{1}{n}\right)\big[1 - \lambda\big(2y + d\left(\frac{3}{n}\right)\big)\big] < \frac{\delta}{1-\delta}\alpha$$

This proposition per se does not mean much in terms of interpretation. However, it induce interesting comparative static properties that we present in Proposition 14.

Proof.

The consumption dictated by the equal consumption rule is $c^i(\theta_0) = \frac{1}{n}\sum y^i(\theta_0)$.

• Weak stability

Weak-stability corresponds to strong punishment.

If $i \in N_j$ and $i \neq o$:

$d_i(g) = n_j - 1$ and $d_i(g_R) = 0$

R is autarky. Thus, $c^i_R(\theta) = \begin{cases} y+d & \text{with probability } p \\ y & \text{with probability } 1-p \end{cases}$

In case of deviation, the best strategy is to deviate to all in order to keep y^i.

Thus, $c^i_D(\theta_0) = y^i(\theta_0)$.

Notice that i deviates if and only if her income realization is $y + d$. And if $0 < m < n$ individuals obtain $y + d$, $F(m) \equiv u(y+d) - u\big(\frac{m(y+d)+(n-m)y}{n}\big) = d(1-\frac{m}{n})\big[1-\lambda\big(2y + d(1+\frac{m}{n})\big)\big]$.

F is decreasing in m and therefore is maximized for $m = 1$. Thus, i has high income realization, and all the other individuals have low income realization.

Furthermore,

$\sum_\theta p(\theta)u\big(c^i_R(\theta)\big) = pu(y+d) + (1-p)u(y) = pd + y - \lambda(y^2 + 2pdy + pd^2)$, and

$\sum_\theta p(\theta)u\big(c^i(\theta)\big) = y + dp - \lambda\big(y^2 + 2dpy + \frac{d^2p}{n}(1+(n-1)p)\big)$.

Thus, the stability condition for i is

$d(1-\frac{m}{n})\big[1-\lambda\big(2y+d(1+\frac{m}{n})\big)\big] < \frac{\delta}{1-\delta}\big\{\lambda\frac{n-1}{n}pd^2(1-p) + \alpha(n_j-1)\big\}$ for all m such that $0 < m < n$.

Hence for $m = 1$, $d(1-\frac{1}{n})\big[1-\lambda\big(2y+d(1+\frac{1}{n})\big)\big] < \frac{\delta}{1-\delta}\big\{\lambda\frac{n-1}{n}pd^2(1-p)+\alpha(n_j-1)\big\}$.

For o, the overlapping individual, $d_o(g) = n-1$. Either $D \subset N_j$ and $d_o(g_R) = n - n_j$, or $D \subset N_1 \cup N_2$ and $d_o(g_R) = 0$.

If $D \subset N_j$, by deviating only to D, o still be with $N_{j'}$. The stability condition is $d(1-\frac{m}{n})\big[1-\lambda\big(2y+d(1+\frac{m}{n})\big)\big] - \epsilon_{j'} - \lambda\big(-2\epsilon_{j'}(y+d)+\epsilon^2_{j'}\big) < \frac{\delta}{1-\delta}\big\{\lambda(\frac{1}{n-n_j+1)} - \frac{1}{n)})pd^2(1-p) + \alpha(n_j-1)\big\}$ for all m such that $0 < m < n$. Where $\epsilon_{j'}$ is the transfer that o makes in the other coalition.

Thus for $m=1$, $\epsilon_{j'}=(n_{j'}-1)\frac{d}{n}=(1-\frac{n_j}{n})d$, and the stability condition for o is $d(1-\frac{1}{n})\big[1-\lambda\big(2y+d(1+\frac{1}{n})\big)\big]-d(1-\frac{n_j}{n})\big[1-\lambda\big(2y+d(1+\frac{n_j}{n})\big)\big]<\frac{\delta}{1-\delta}\big\{\lambda(\frac{1}{n-n_j+1)}-\frac{1}{n)})pd^2(1-p)+\alpha(n_j-1)\big\}$, what is equivalent to

$d(\frac{n_j}{n}-\frac{1}{n})\big[1-\lambda\big(2y+d(\frac{n_j}{n}+\frac{1}{n})\big)\big]<\frac{\delta}{1-\delta}\big\{\lambda(\frac{1}{n-n_j+1}-\frac{1}{n})pd^2(1-p)+\alpha(n_j-1)\big\}$.

If $D\subset N_1\cup N_2$, the stability condition is

$d(1-\frac{m}{n})\big[1-\lambda\big(2y+d(1+\frac{m}{n})\big)\big]<\frac{\delta}{1-\delta}\big\{\lambda\frac{n-1}{n}pd^2(1-p)+\alpha(n-1)\big\}$ for all m such that $0<m<n$.

Thus, for $m=1$, $d(1-\frac{1}{n})\big[1-\lambda\big(2y+d(1+\frac{1}{n})\big)\big]<\frac{\delta}{1-\delta}\big\{\lambda\frac{n-1}{n}pd^2(1-p)+\alpha(n-1)\big\}$.

To summarize, weak-stability is equivalent to

(i) $d(1-\frac{1}{n})\Big[1-\lambda\big(2y+d(1+\frac{1}{n})\big)\Big]<\frac{\delta}{1-\delta}\Big\{\lambda\frac{n-1}{n}d^2p(1-p)+\alpha(n_j-1)\Big\}$

(ii) $d(\frac{n_j}{n}-\frac{1}{n})\Big[1-\lambda\big(2y+d(\frac{n_j}{n}+\frac{1}{n})\big)\Big]<\frac{\delta}{1-\delta}\Big\{\lambda(\frac{1}{n-n_j+1}-\frac{1}{n})d^2p(1-p)+\alpha(n_j-1)\Big\}$

(iii) $d(1-\frac{1}{n})\Big[1-\lambda\big(2y+d(1+\frac{1}{n})\big)\Big]<\frac{\delta}{1-\delta}\Big\{\lambda\frac{n-1}{n}d^2p(1-p)+\alpha(n-1)\Big\}$

Notice that $n>n_j$. Therefore, (i) implies (iii).

• Strong stability

For strong-stability, either $i\neq o$ deviate to n_j-1 others and stays alone (corresponding to strong punishment) or deviates to $0<l<n_j-1$ others except o and still be in connection the whole community (so i will not face the direct effect of punishment). As for o, either she deviates to $n-1$ or to n_j-1(corresponding to strong punishment), or she deviates to $0<l<n-1$ (with $D\subset N_1\cup N_2$) and she is still linked with the whole community.

As we pointed out earlier, the state where the deviation gain is maximal corresponds to $m=1$. Thus, strong-stability condition is

$d(1-\frac{1}{n})\big[1-\lambda\big(2y+d(1+\frac{1}{n})\big)\big]-\epsilon_l-\lambda\big(-2\epsilon_l(y+d)+\epsilon_l^2\big)<\frac{\delta}{1-\delta}\alpha l.$

Where ϵ_l is the transfer that the deviator fails to make to l individuals. In the state where $m=1$, $\epsilon_l=\frac{n-1-l}{n}d$ (because if the transfer to o is not made, $i\neq o$ will not benefit from the consumption smoothing) for $i\neq o$, and $\epsilon_l=\frac{n-1-l}{n}d$ for o.

Thus strong-stability is equivalent to the weak-stability conditions plus

$d(\frac{l}{n})\big[1-\lambda\big(2y+d(\frac{l+2}{n})\big)\big] < \frac{\delta}{1-\delta}\alpha l$. But $l>0$, so $d(\frac{1}{n})\big[1-\lambda\big(2y+d(\frac{l+2}{n})\big)\big] < \frac{\delta}{1-\delta}\alpha$. The LHS is a decreasing function of l. For all l the condition must be verified, then it must be the case for the maximum value of LHS. This value is obtained with the minimum value of l which is $l=1$. Thus, the stability condition is $d(\frac{1}{n})\big[1-\lambda\big(2y+d(\frac{3}{n})\big)\big] < \frac{\delta}{1-\delta}\alpha$. □

We remind the reader that set of parameters we investigate is $\mathcal{P} \equiv \{y, d, p, \lambda, \alpha, n_1, n_2\}$. In the sequel, for each $x \in \mathcal{P}$, we use the notation (IIO_g, x_k) to designate the IIO_g where, everything equal, the parameter x takes the value x_k.

Proposition 13. ***Comparative statics***

a) ***Income***

Let $y \equiv y_1$ and $y \equiv y_2$ be two values of the low income realization such that $y_2 > y_1$. If (IIO_g, y_1) is stable, then (IIO_g, y_2) is also stable.

b) ***Externalities***

Let $\alpha \equiv \alpha_1$ and $\alpha \equiv \alpha_2$ be two values of the externality function coefficient such that $\alpha_2 > \alpha_1$.

If (IIO_g, α_1) is stable, then (IIO_g, α_2) is also stable.

c) ***Probability of income realization***

Let $p \equiv p_1$ and $p \equiv p_2$ be two values of the probability of realization of the high income such that $p_2 > p_1$.

- *For $0.5 > p_2 > p_1$, if (IIO_g, p_1) is stable, then (IIO_g, p_2) is also stable.*
- *For $p_2 > p_1 > 0.5$, if (IIO_g, p_2) is stable, then (IIO_g, p_1) is also stable.*

d) ***Risk aversion***

Let $\lambda \equiv \lambda_1$ and $\lambda \equiv \lambda_2$ be two values of the risk aversion coefficient such that $\lambda_2 > \lambda_1$. If (IIO_g, λ_1) is stable, then (IIO_g, λ_2) is also stable.

e) ***Size of coalition***

Let $n_j \equiv n_j^1$ and $n_j \equiv n_j^2$ be two values of the size of the coalition N_j, $j = 1, 2$ such *that* $n_j^2 > n_j^1$ *with* n *constant.*

If (IIO_g, n_j^1) *is stable, then* (IIO_g, n_j^2) *is also stable.*

Interpretation.

a) Everything equal, if an IIO with low income is stable, it remains stable if the income level increases. A stable IIO does not break up if the community becomes more wealthy, as far as the dispersion in income realization is the same.

b) Everything equal, if an IIO with low external cost, then if this cost is higher, the IIO remains stable. An increase in the external cost of losing a link reinforce stability.

c) Everything equal, for low probabilities, an increase in the probability of realization of the highest income preserves stability. On the other hand, a decrease in the probability of realization of the highest income preserves stability as long as this probability is high enough.

d) Everything equal, an increase in risk aversion preserves stability. This is very intuitive because as individuals become more risk averse, they will like to smooth more future consumption and therefore not deviate today.

e) Everything equal, if the size of one of the coalition is increasing such that the number of overall individuals is unchanged, then an increase in the coalitional size preserves stability.

Proof.

a) All the stability conditions can be written as $A[1 - \lambda(2y + B)] < C$, where $A > 0$, and $\lambda > 0$. Therefore, $y_2 > y_1 \Rightarrow A[1 - \lambda(2y_2 + B)] < A[1 - \lambda(2y_1 + B)]$

b) All the stability conditions can be written as $A < B(C + \alpha D)$, where $B > 0$, and $D > 0$. Therefore, $\alpha_2 > \alpha_1 \Rightarrow B(C + \alpha_2 D) > B(C + \alpha_1 D)$.

c) All the stability conditions (i) can be written as $A < Bp(1-p) + C$, where $B > 0$. But the functional $F(p) = p(1-p)$ defined on $(0, 1)$, increases in $(0, 0.5)$ and decreases in $(0.5, 1)$. The probability p is irrelevant for condition (ii).

d) All the stability conditions can be written as $A - \lambda B < \lambda C + D$, where $B > 0$, and $C \geq 0$. We can rewrite the stability conditions as $-\lambda(B + C) < D - A$, where $B + C > 0$. Therefore, $\lambda_2 > \lambda_1 \Rightarrow -\lambda_2(B + C) < -\lambda_1(B + C)$.

e) There are three conditions of stability. For the first conditions (i) (with $k = n$), the result is straightforward because the coefficient of n_j is positive. For $k = n_j$ in condition (i),

$\frac{d}{n}\Big[1 - \lambda\big(2y + d(\frac{n_j}{n} + \frac{1}{n})\big)\Big] < \frac{\delta}{1-\delta}\Big\{\lambda(\frac{1}{n(n-n_j+1)})d^2p(1-p) + \alpha\Big\}$ because $n_j - 1 > 0$.

This result can be generalized as $\frac{A}{n(n-n_j+1)} + Bn_j > C$, where $A > 0$ and $B > 0$. The result obtains because the LHS is an increasing function of n_j. $\square$

4.4.2 Numerical simulations

In the previous section we investigate how exogenous parameters affect stability. However when combined, the magnitude of the effect of each such parameters on stability is ambiguous. Therefore, we proceed by simulation to compute these effects.

In the following, we simulate our simple model of IIO and we compute stability with non extreme values of the discount factor.

For each value of δ (on a grid), we verify stability conditions for several values of each given parameter $x \in \mathcal{P}$ and we compute the proportion of stable (IIO_g, x).

For weak stability, Figure 4.3 shows that two parameters tend to have no ambiguous effects. The level of output tend to be the most pertinent parameter that reinforce stability whereas the size of coalition tend to be the least.[21] Other parameters in consideration have intermediate effect on stability. Ranking them from the most to the least pertinent, we have approximately : external cost, risk aversion, and difference in output. One effect that we identify (but not observable from Figure 4.3) is that even if the size of coalitions tend to be less pertinent for stability, as the dis-

21. These results are obtained for the discount factor ranging from 0.4 to 0.6

count becomes very high, weak-stable IIOs are characterized by two coalitions, with disproportionate sizes. The same result is obtained by Bramoullé and Kranton (2007) in the bilateral setting.

For the same values of parameters as for weak stability, Figure 4.4 shows that it is more difficult to achieve strong stability.[22] This result is very intuitive because the more severe the punishment, the less the incentive to deviate. The most pertinent parameter for stability tend to be the difference in output. It is followed by income level, external cost, and risk aversion.

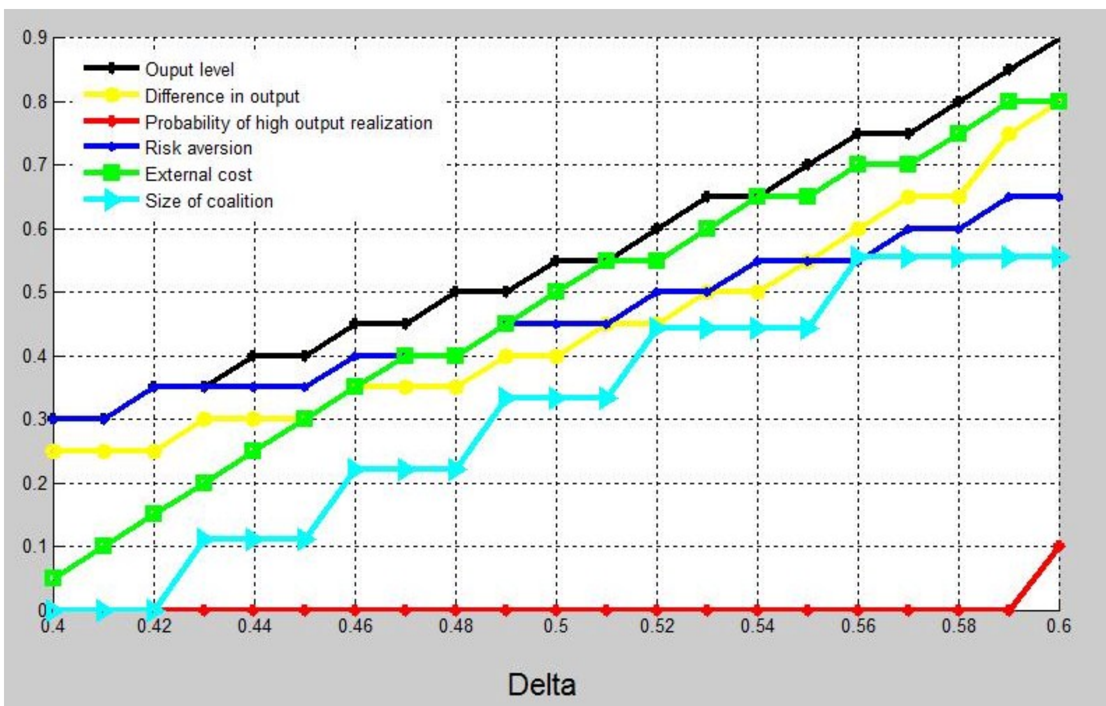

Figure 4.3 – Weak-stability and exogenous parameters

22. Strong stability is hard to obtain for lower values of δ. We do not use the same range of δ in Figure 4.4 because for lower values, almost all IIOs are unstable. Here we use a ranging from 0.5 to 0.7

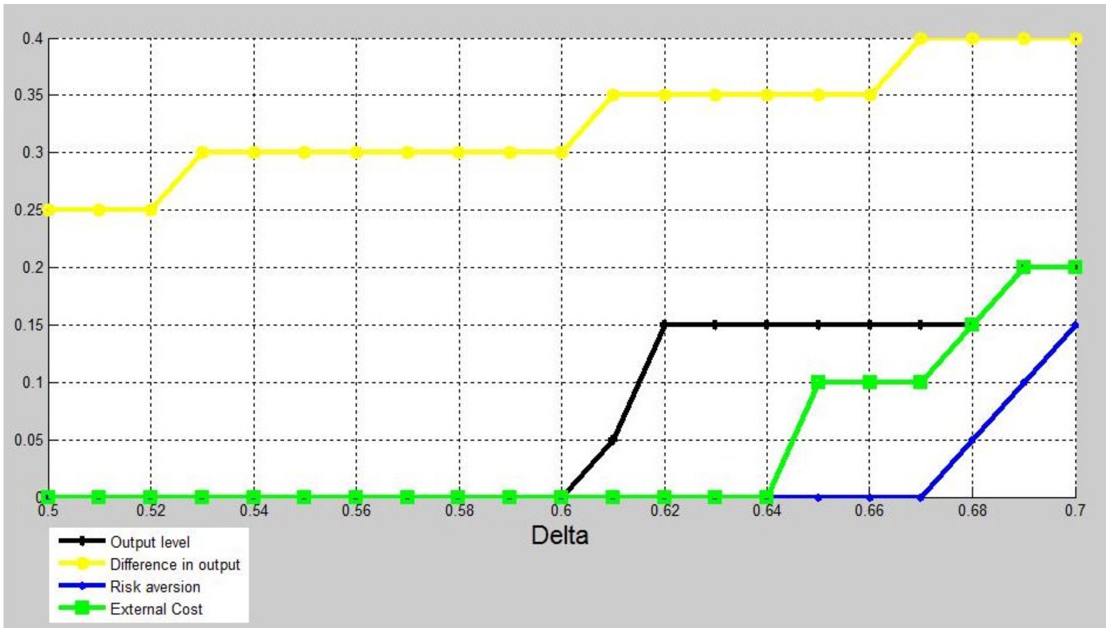

FIGURE 4.4 – Strong-stability and exogenous parameters

4.5 Example of full risk sharing

In this chapter, we use equal sharing rule at component level, obtained by equal consumption in coalitions. Since this is a strong assumption, we develop a procedure that can help to implement this multilateral sharing rule in reality.

The sharing rule specifies the aggregate transfers for all individuals, conditional on the distribution of realized incomes. But, for a given distribution of aggregate transfers, there exist multiple ways to specify the detailed transfer flow between individuals. For example in the bilateral setting, Bramoullé and Kranton (2007) show that if pairs of individuals meet randomly and share equally their income, and if *the number of meeting rounds tends to infinity*, then equal consumption obtains for all linked individuals. This is a very interesting result. However, it is difficult to implement because of the infinite number of rounds needed. Suppose that a policy maker intends to attain full risk pooling in a community of linked individuals. It is not obvious how to implement infinite rounds of meetings between two harvests. Furthermore in our setting, as transfers across networks can only be performed by overlapping individuals, the implementation of infinite round meetings is much more complicated.

In this section, we develop a *finite rounds* transfer scheme that equates consumption for linked individuals. We call this transfer scheme the *ordered proportional transfer scheme*. It has both descriptive and normative aspects. First, this transfer scheme is based on observations made in rural communities in developing countries. If there is a shock in the net income of an individual i, groups which are directly linked to i make transfers to her according to their own wealth such that at the end of the day i receives the needed transfer. Due to this procedure, individual income shocks does not hardly affect consumption. For example that pattern is observed in Tanzanian villages for income shocks due to illness or ceremonies (De Weerdt and Dercon, 2005). Second, this transfer scheme induces full risk pooling for all linked individuals.

Consider a population N of n individuals, a preexisting network g, the induced IIO_g, a coalition $S \in \gamma_g$, and a vector of realized income y for all individuals in N. According to Proposition 11, the equal consumption rule prescribes that each individual in N consumes $\overline{c} \equiv \frac{1}{n}\sum_{i\in N} y^i$.[23] Thus for all $i \in S$, the overall transfers is $z^i = \overline{c} - y^i$. If furthermore i is an overlapping individual, then z^i_S denotes i's net transfer within the coalition S. For all $i \in S$, let z^{il}_S denote the detailed transfer that individual i makes to a coalitional fellow $l \in S$. Thus, $\sum_{l\in S} z^{il}_S = z^i_S$. Let $E_S \equiv \{i \in S \mid z^i_S < 0\}$ denote the set of individuals who have an excess of income in S compared to $\overline{c}$.

Definition 25. *The proportional transfer scheme is such that :*

(i) If $i \in E_S$, then $z^{il}_S = -\frac{z^i_S}{\sum_{k\in E_S} z^k_S} z^l_S$ for all $l \notin E_S$.

(ii) If $i \notin E_S$, then $z^{il}_S = -z^{li}_S$ for all $l \in E_S$.

(iii) Otherwise, $z^{il}_S = 0$.

23. Remember that we limit ourselves to the case where all individuals are linked directly or indirectly.

Remark 22. The proportional transfer states that only individuals with excess in income can make transfers. These transfers are made, proportionally to their needs (in order to obtain the prescribed consumption $\bar{c}$), to those who suffer a lack of income. If the realized income of an individual is $\bar{c}$, then she neither makes nor receives any transfer.
If g is the complete network, then $\gamma_g = \{N\}$, and the *ordered proportional transfer* coincides with the proportional transfer.

Consumption can be equalized for the whole N only because of overlapping individuals. They are the only channel through which transfers can be made across coalition. They serve as wealth spreader from the wealthiest (highest per capita) coalitions to the poorest (lowest per capita) coalitions that they belong to. Therefore, overlapping individuals are in charge not only for their own needs but also for the needs of the poor coalitions that they belong to. This is possible because we made the assumption that within coalitions, each individual can observe the aggregated transfers of her coalitional fellows, especially overlapping individuals.

The *ordered proportional transfer scheme* consists of the following algorithm. It begins once the distribution of the realized income is known (for example after each harvest). At first, we need to specify the net transfer for overlapping individuals in each coalition. A simple way to do this is to rank coalitions from the lowest per capita to the highest per capita. In each coalition where the per capita realized income is less than $\bar{c}$, only the overlapping individuals can satisfy the needs. Therefore by applying the proportional transfer within each coalition, one can consistently determine the transfer from each overlapping. In case of tie, share equally the needed transfer among overlapping individuals. In the following, we develop the algorithm.
At the beginning, let $\gamma \equiv \gamma_g$.

Step 1 : Rank all the coalitions in γ according to per capita income realization.

Let S^* denote the coalition with the highest per capita income realization. [24]

24. In case of ties, choose the coalition that contains the highest indexed individual.

Step 2 : For each overlapping individual $i \in S^*$, replace y^i with $y'^i \equiv y^i + z^i - z^i_{S^*}$ and implement the proportional transfer scheme in S^*. The value y'^i represents the aggregated needs of i.[25] In practice, it can easily be obtained by proceeding backwards from the poorest to the richest coalitions containing i.

Step 3 : Update the income of each overlapping individual in S^* to $y^i_{updated} \equiv y^i + z^i_{S^*}$. Update γ to $\gamma \setminus \{S^*\}$ by removing S^* and go to Step 1.

The algorithm ends if $\gamma = \emptyset$.

Let m^* be the number of distinct coalitions in γ_g.

Theorem 13. *The ordered proportional transfer algorithm is well defined and equates consumption in γ_g after m^* rounds.*

Before we prove the theorem, the following example illustrates the ordered proportional transfer for a 5-person IIO.

Example 8. Consider a community $N = \{A, B, C, D, E\}$ of 5 individuals, and a preexisting network g linking them. In brackets are the realized incomes.

In this example, $N = \{A, B, C, D, E\}$. The coalitions are $S_1 = \{A, B, C\}, S_2 = \{C, D\}, S_3 = \{B, E\}$. The IIO is $\gamma_g = \{S_1, S_2, S_3\}$. Individual realized incomes are $y^A = 10, y^B = 2, y^C = 2, y^D = 0, y^E = 1$. The average consumption is $\bar{c} = 3$. For all $i \in N$, the aggregated transfer is $z^i = \bar{c} - y^i$. Thus, $z^A = -7, z^B = 1, z^C = 1, z^D = 3, z^E = 2$. There are two overlapping individual : B and C. From the lowest to the highest per capita realized income, coalitions are ranked is this order : S_1, S_3, S_2. The coalitional aggregated transfers for overlapping individuals are : $Z^B_{S_3} = -2, Z^C_{S_2} = -3$. These transfers are to be made to E and D so that they consume the prescribed amount.

Round 1 : $\gamma = \{S_1, S_2, S_3\}$

The highest per capita realized income coalition is $S^* = S_1$. The new income distribu-

25. This value can be negative. Furthermore, remember that each individual in S^* can observe i's aggregate transfer.

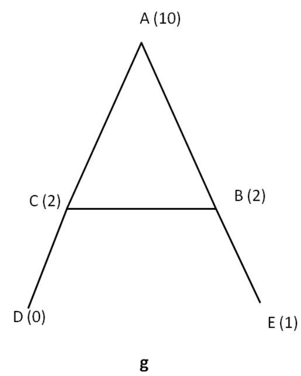

FIGURE 4.5 – An example of a 5-person network

tion for the overlapping individual B is $y'^B = y^B + z^B - z^B_{S^*} = y^B + Z^B_{S_3} = 2 - 2 = 0$, and for individual C is $y'^C = y^C + z^C - z^C_{S^*} = y^C + Z^C_{S_2} = 2 - 3 = -1$. According to the proportional transfer, A ends up with 3, B receives 3 and C receives 4. The updated $\gamma = \{S_2, S_3\}$. The updated incomes are $y^B_{updated} = 2 + 3 = 5, y^C_{updated} = 2 + 4 = 6$

Round 2 : $\gamma = \{S_2, S_3\}$ with $y^B = 5, y^C = 6, y^D = 0, Y^E = 1$

The highest per capita realized income coalition is $S^* = S_3$. There is no overlapping individual. The income distribution is $y^B = 5, y^E = 1$. According to the proportional transfer, B ends up with 3, and E with 3. The updated $\gamma = \{S_2\}$.

Round 3 : $\gamma = \{S_2\}$ with $y^C = 6, y^D = 0$

The highest per capita realized income coalition is $S^* = S_2$. The new income distribution is $y^C = 6, y^D = 0$. According to the proportional transfer, C ends up with 3, and D with 3. The updated $\gamma = \emptyset$.

The algorithm ends after 3 rounds with $c^A = c^B = c^C = c^D = c^E = 3$.

Proof of the Theorem.

At each round, S^* is selected and following the proportional transfer, each non-overlapping individuals in S^* ends up with $\bar{c}$. The excess of realized income are transferred to the overlapping individuals proportionally to their aggregated needs.

At the first round, the total amount of available income in the economy is $Y_1 = n\bar{c}$.

We partition S^* into $S^* = N' \cup O$, where N' is the subset of non-overlapping players, of size n', and O is the subset of overlapping players, of size o. At Step 3, $\sum_{i\in O} y^i_{updated} = \sum_{i\in S^*} y^i - n'\overline{c}$, as each non-overlapping individual in S^* consumes $\overline{c}$.

At round 2, the total amount of available income in the economy is $Y_2 = \sum_{i\in N-S^*} y^i + \sum_{i\in S^*} y^i - n'\overline{c}$. Hence, $Y_2 = \sum_{i\in N} y^i - n'\overline{c} = (n-n')\overline{c}$. Therefore, $\frac{1}{(n-n')}Y_2 = \overline{c}$. So, at the beginning of round 2, the problem is the same as at round 1 but with $n - n'$ individuals. At round m, the last one, the available overall income is, for the same justifications, $Y_m = k\overline{c}$ to be distributed among k individuals according to the proportional transfer scheme. □

As also obtained by Bramoullé and Kranton (2007), our transfer scheme equates consumption for all linked individuals, but in finite steps. One difference is however, in Bramoullé and Kranton (2007) knowledge of y and z is not required. Moreover, the algorithm is easy to implement. Besides, it is worth noting that the role of overlapping individuals in the perpetuation of transfers across coalitions is essential for the IIO to fully pool risk.

4.6 Discussion and conclusion

In this chapter, we take a step forward in the modeling of informal insurance. We depart from the traditional extreme approaches using groups and the networks. As a result, we characterize stable informal insurance arrangements without imposing limits on discounting. This chapter is a first step in the exploration of a multilateral approach to model informal insurance. Since the economics literature on the topic is very rich, we discuss in this section other existing aspects in the literature that can enrich our approach and therefore can be subject to future investigations.

First, our notion of stability implicitly models economic agents as being myopic.

But in real life, when contemplating a deviation, the deviator may not only consider the immediate effect of her action, but also future deviations that may follow. At least one should require the next network to form following a deviation to be itself immune to future deviations. This consistency notion induces a recursive definition of stability as done by BGR. Besides, we explore only individual deviations. But a group of individuals may also deviate in order to stay together in the future as studied by Genicot and Ray (2002). We choose the simplest way to model economic agents in this setting because we want to address an approach that merges groups and bilateral links approaches as a first step in a new model of informal insurance.

Second, most of the papers in this literature answer the following questions. What is the best transfer in order to obtain a Pareto efficient risk sharing ? If we cannot reach Pareto efficiency, what is the transfer that guarantee Pareto frontier, conditional on self enforcing constraints ? These questions are widely answered in the literature and we do not intend to contribute in that direction. This chapter answers instead the following question. Suppose that individuals in isolated islands implement the first best with respect to their norms. [26] *What are the determinants of the preexisting network such that an IIO perpetuates transfers over time ?*

Third, we investigate strong and weak punishments. It will be interesting to explore other punishment schemes. The range between the weak and the strong punishment is due to the modeling of the reaction of *third parties*. These are individuals who are not directly victims of a deviation, but are linked to the victim and therefore are aware of this deviation. As an alternative to these extreme punishments, an intermediary case is developed by BGR, the level-q punishment. *Following a deviation, all individuals who are connected to a victim by a path not exceeding length q (but not via deviant) sever direct links (if any) with the deviant.* The problem with level-q punishment is that the decision by a third party to severe a link or not is dictated exogenously by the number q. In our setting, the intermediary level-q punishment

26. It is documented in empirical findings that in a community, individuals in isolated islands implement the first best with respect to their norms.

is either the weak or the strong punishment.[27] An intermediary punishment can be obtained with third party individuals severing the link with the deviator with some probability. If this probability equals 1, the strong punishment applies and if it equals 0, the weak punishment applies.

Finally, it is useful to explore other sharing norms. For example one can consider sharing rules such that individuals keep some part of the realized income and share the remaining with coalitional fellows. These sharing rules can be viewed as extensions of equal consumption rule that we explore in this chapter.

In this chapter we develop a model of informal insurance organizations that accommodates stylized facts. These are the facts that informal insurance organizations are homogenous groups of individuals, these groups may overlap, and there exists externalities across them. For this purpose, we use multilateral norms that merge the group oriented and the network oriented approaches. We provide a transfer scheme that permits to equate consumption for linked individuals in finite rounds. We characterize stability under weak and strong punishments. Finally we provide characteristics of some exogenous parameters of the model regarding stability.

27. First, the level-0 punishment is the weak punishment. Besides, within coalitions, all individuals are directly linked. So, there is no path exceeding one that links all individuals within a coalition. Therefore the strong punishment coincide with the level-1 punishment.

Conclusion générale

Cette thèse est un recueil de quatre articles traitant des coalitions imbriquées. Elle propose non seulement des éléments de réponses aux questions relatives au processus de formation de ces coalitions, mais aussi utilise ces réponses pour proposer des modèles d'assurance informelle (qui sont des coalitions imbriquées) dont la construction est basée sur des faits empiriques.

Dans le premier chapitre, je propose un jeu dynamique dont l'équilibre donne une structure de coalitions imbriquées. Je détermine ensuite les conditions d'existence d'équilibre et je propose une procédure permettant de calculer l'équilibre dans le cas des jeux symétriques. J'établis ensuite un lien biunivoque entre coalitions imbriquées et réseaux interrelationnels et enfin je caractérise la plupart des structures de réseaux comme émanant d'un équilibre du jeu dynamique proposé.

Dans le deuxième chapitre, je procède par une approche par les jeux de coopération pour expliquer la formation des coalitions imbriquées. Il s'agit d'un processus où la distribution du gain collectif obtenu au sein des coalitions est un élément du noyau. Je propose donc une extension du noyau récursif au cas où les coalitions sont imbriquées. Je propose deux cas : un premier cas où l'extension est naïve et présente d'énormes restrictions et un deuxième cas où je tiens compte des sophistications liées au caractère imbriqué des coalitions (notamment du caractère particulier des individus appartenant à plusieurs coalitions). J'obtiens ainsi une structure de noyau plus raffinée et plus en harmonie avec les coalitions imbriquées.

Dans le troisième chapitre, j'apporte des éléments nouveaux à la modélisation des assurances informelles : la prise en compte des externalités, les coalitions imbriquées et un processus d'autodiscipline endogène. Sur cette base, je caractérise les groupes d'assurance informelle stables et j'isole les paramètres clés qui peuvent inciter les agents à faire défection. Je dérive enfin des résultats de statiques comparatives pour ces paramètres.

Enfin dans le dernier chapitre, je propose un modèle d'assurance informelle qui inclut les faits empiriques qui existent dans la littérature. J'observe ce qu'il advient dans les deux cas extrêmes d'autodiscipline. Je caractérise la stabilité des groupes d'assurance informelle sans faire de restrictions sur le facteur d'escompte. J'identifie ensuite les caractéristiques des réseaux relationnelles qui contribuent à la stabilité des groupes d'assurance informelle. Enfin j'isole un certain nombre de paramètres clés qui affectent la stabilité et je fournis des résultats de statiques comparatives pour ces paramètres. Quelques simulations me permettent de classer ces paramètres suivant leur contribution à la stabilité. Enfin je propose une procédure plus réaliste qui permet de lisser au mieux la consommation au sein des groupes d'assurance informelle.

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